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ELEMENTS AND PARAMETERS OF THE OSCULATING ORBIT AND THEIR DERIVATIVES

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SUMMARY

The analysis determines the time derivatives of the conic-section orbital elements in a disturbed orbit by perturbation methods. Integration of any of the several resulting systems of six first-order linear differential equations by numerical methods can be a useful tool for the solution of problems in orbital mechanics.

Equations for the two-body orbit are also summarized in a convenient manner.

INTRODUCTION

The recent emphasis on space-flight trajectory calculations and the use of electronic computing machinery have combined to increase the interest in the perturbation methods for studying problems in celestial mechanics. This is especially true of the methods that leave the disturbing function undeveloped and require numerical integration to complete the solution. Previously, the amount of numerical work required rendered precision calculations by these latter methods impractical. It is the purpose of this report to examine and extend some of the previous work in perturbation theory to secure forms that may be better adapted to numerical integration, at least for specific problems.

The basic work in developing expressions for the derivatives of the orbital elements must be credited to various notable contributors in dynamical astronomy. Perturbation theory was begun by Euler in 1748. However, the first complete development was presented by Lagrange in 1782.

The perturbation method summarized herein is formulated in terms of Lagrangian brackets. Numerous methods for evaluating the brackets have been published. The indirect method of evaluating the brackets used herein is attributed to Whittaker, as reported by Smart (ref. 1). The characteristic of the indirect methods is that the work begins with the derivation of a general expression for a Lagrangian bracket, from which all brackets are easily evaluated.

For convenience, the present report gives alternative forms of the perturbation derivatives; and, by using the results presented in the tables, many others are obtainable. The extension to the case of circular orbits has been included. A collection of useful two-body equations is also given in table I without derivation.

The procedure indicated for the reduction of the three second-order linear differential equations of motion in rectangular coordinates to six first-order linear differential equations in orbital elements follows the pattern, but is revised from that given in Moulton (ref. 2). The analysis has been further generalized by avoiding the requirement that the perturbation function be a potential function. This extension shows that the results are valid for thrust and drag, which are not potential-type functions. Another revision concerns the determination of the disturbing functions in terms of the elements that define the size, shape, and position in the orbit. This procedure given herein is believed to be more direct than that given elsewhere (refs. 1 and 2).

ANALYSIS

Equations of Motion

Consider the motion of an object subject to an inverse-square central gravitational acceleration directed toward the origin, and also subject to smaller disturbing accelerations that can be expressed as functions of the variables and constants of the problem. Let OX, OY, OZ in figure 1 be the coordinate axes in a noninertial Cartesian system having its origin located at the center of the mass M_o. The equations of motion of the object are then as follows from application of Newton's second law to the problem (e.g., refs. 1 and 2):

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{\mu x}{r^3} &= A_x \\ \frac{d^2y}{dt^2} + \frac{\mu y}{r^3} &= A_y \\ \frac{d^2z}{dt^2} + \frac{\mu z}{r^3} &= A_z \end{aligned}\right\} \tag{1}$$

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In the notation adopted hereinafter, equation (1) will be indicated as

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = A_x, \qquad x \to y,z \tag{1}$$

where $x \to y,z$ indicates that separate equations in x, y, and z are included to complete the set. All symbols are defined in appendix A. The acceleration term $\mu x/r^3$ is due to central gravitational attraction, where

$$\mu = k^2(M_o + m) \tag{2}$$

(k^2 is the gravitational constant, M_o is the mass of the body at the origin, and m is the object's mass). The components of the disturbing acceleration that disturb the two-body orbit are A_x, $x \to y,z$ in equations (1) and may be written as

$$A_x = f_x - k^2 \sum_{i=1}^{n} M_i\left(\frac{x - x_i}{\Delta_i^3} + \frac{x_i}{r_i^3}\right), \qquad x \to y,z \tag{3}$$

where M_i is the mass of the i^{th} disturbing body, n is the number of gravitating bodies excluding the central body, Δ_i is defined by

$$\Delta_i^2 = (x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 \tag{4}$$

and f_x is the component of the disturbing acceleration along OX due to all other forces. For example, these may include propulsion thrust, aerodynamic forces, and forces due to the oblateness of M_o. No restriction need be placed on the form of disturbing acceleration except that it be sufficiently well defined to permit integration.

Equations (1) may be integrated directly by numerical methods. However, in many cases larger intervals may be used in numerical integration, or approximate closed-form solutions can be obtained if the equations are expressed in terms of perturbations of orbital elements.

The perturbation theory uses as a reference an orbit having no perturbations. If the disturbing acceleration is assumed to be zero, the differential equations become

$$\frac{d^2x}{dt^2} + \frac{\mu x}{r^3} = 0, \qquad x \to y,z \tag{5}$$

The solution of equations (5) is readily obtained and is found to be a conic section. The motion of the object can be represented by six orbital elements obtained from the constants of integration of equations (5). Table I gives a collection of two-body equations relating selected orbital elements and parameters. Although it is not possible to express the Cartesian positions and velocities explicitly in terms of the orbital elements, the solutions of equations (5) may be indicated as

$$x = x(c_1, c_2, c_3, c_4, c_5, c_6, t), \qquad x \rightarrow y,z \tag{6a}$$

$$\dot{x} = \dot{x}(c_1, c_2, c_3, c_4, c_5, c_6, t), \qquad x \rightarrow y,z \tag{6b}$$

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where $c_1, c_2, \cdot \cdot \cdot, c_6$ are orbital elements.

If $A_x \neq 0$, $x \rightarrow y,z$ in equations (1), the path is not a simple conic. However, at any instant it may be regarded as a conic with variable orbital elements. In fact, equations (6) are the solutions of equations (1) if the orbital elements $c_1, c_2, \cdot \cdot \cdot, c_6$ are regarded as variables.

This introduces the concept of the osculating orbit. Let an object be moving in a perturbed path about a central body. An instantaneous two-body orbit always exists tangent to the actual path at the point and having a velocity in the orbit equal to that of the actual body. Such tangent orbits are called the osculating orbits. The relations implicit in this definition are used to derive the equations for the disturbed orbit in terms of orbital elements. The three second-order differential equations of equations (1) are transformed to six first-order simultaneous differential equations involving orbital elements. Lagrangian brackets are utilized to solve the set of simultaneous differential equations for derivatives of the various orbital elements explicitly. The explicit derivatives are the objective of this report.

As the first step in obtaining these derivatives, equations (6) are differentiated regarding $c_1, c_2, \cdot \cdot \cdot, c_6$ as variables. The following equations that apply to the actual path are obtained:

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \sum_{k=1}^{6} \frac{\partial x}{\partial c_k} \frac{dc_k}{dt}, \qquad x \rightarrow y,z \tag{7a}$$

$$\frac{d^2x}{dt^2} = \frac{\partial \dot{x}}{\partial t} + \sum_{k=1}^{6} \frac{\partial \dot{x}}{\partial c_k} \frac{dc_k}{dt}, \qquad x \rightarrow y,z \tag{7b}$$

Equations (1) may be introduced to eliminate (d^2x/dt^2), $x \rightarrow y,z$ from equations (7b). Then equations (7) may be written as

$$\frac{\partial x}{\partial t} - \frac{dx}{dt} + \sum_{k=1}^{6} \frac{\partial x}{\partial c_k} \frac{dc_k}{dt} = 0, \qquad x \rightarrow y,z \tag{8a}$$

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$$\frac{\partial \dot{x}}{\partial t} + \frac{\mu x}{r^3} + \sum_{k=1}^{6} \frac{\partial \dot{x}}{\partial c_k} \frac{dc_k}{dt} = A_x, \qquad x \rightarrow y,z \tag{8b}$$

In the osculating orbit $dc_k/dt = 0$ and $A_x = 0$, so that equations (8) become

$$\frac{\partial x}{\partial t} - \frac{dx}{dt} = 0, \qquad x \rightarrow y,z \tag{9a}$$

$$\frac{\partial \dot{x}}{\partial t} + \frac{\mu x}{r^3} = 0, \qquad x \rightarrow y,z \tag{9b}$$

Introducing the requirement that velocities in the actual path and in the osculating orbit are equal, equations (9a) may be substituted into equations (8a); and similarly, because the acceleration in the osculating orbit differs from that of the true orbit only by the disturbing acceleration, equations (9b) may be substituted into equation (8b) to give

$$\sum_{k=1}^{6} \frac{\partial x}{\partial c_k} \dot{c}_k = 0, \qquad x \rightarrow y,z \tag{10a}$$

$$\sum_{k=1}^{6} \frac{\partial \dot{x}}{\partial c_k} \dot{c}_k - A_x = 0, \qquad x \rightarrow y,z \tag{10b}$$

Equations (10) are the resulting six first-order differential equations. They are not adapted for computation because equations (6) are not explicitly available and because the derivatives of the orbital elements appear simultaneously rather than explicitly. This difficulty is conveniently removed by further manipulation. The following equations are written in a form convenient for formulation in terms of Lagrangian brackets:

$$\left[\left(\sum_{k=1}^{6}\frac{\partial\dot{x}}{\partial c_k}\dot{c}_k - A_x\right)\frac{\partial x}{\partial c_j} - \left(\sum_{k=1}^{6}\frac{\partial x}{\partial c_k}\dot{c}_k\right)\frac{\partial\dot{x}}{\partial c_j}\right.$$

$$+\left(\sum_{k=1}^{6}\frac{\partial\dot{y}}{\partial c_k}\dot{c}_k - A_y\right)\frac{\partial y}{\partial c_j} - \left(\sum_{k=1}^{6}\frac{\partial y}{\partial c_k}\dot{c}_k\right)\frac{\partial\dot{y}}{\partial c_j}$$

$$\left.+\left(\sum_{k=1}^{6}\frac{\partial\dot{z}}{\partial c_k}\dot{c}_k - A_z\right)\frac{\partial z}{\partial c_j} - \left(\sum_{k=1}^{6}\frac{\partial z}{\partial c_k}\dot{c}_k\right)\frac{\partial\dot{z}}{\partial c_j}\right]_{j=1,2,\cdots,6} = 0 \qquad \text{(11a)}$$

The validity of equations (11a) is obvious because each term contains a zero factor from equations (10). Results of the operations shown may be written as

$$\sum_{k=1}^{6}[c_j,c_k]\dot{c}_k = D_{c_j}, \qquad j = 1,2,\cdot\cdot\cdot,6 \qquad \text{(11b)}$$

where

$$[c_j,c_k] = \frac{\partial x}{\partial c_j}\frac{\partial\dot{x}}{\partial c_k} - \frac{\partial\dot{x}}{\partial c_j}\frac{\partial x}{\partial c_k} + \frac{\partial y}{\partial c_j}\frac{\partial\dot{y}}{\partial c_k} - \frac{\partial\dot{y}}{\partial c_j}\frac{\partial y}{\partial c_k} + \frac{\partial z}{\partial c_j}\frac{\partial\dot{z}}{\partial c_k} - \frac{\partial\dot{z}}{\partial c_j}\frac{\partial z}{\partial c_k} \qquad \text{(12)}$$

and

$$D_{c_j} = A_x\frac{\partial x}{\partial c_j} + A_y\frac{\partial y}{\partial c_j} + A_z\frac{\partial z}{\partial c_j}, \qquad j = 1,2,\cdot\cdot\cdot,6 \qquad \text{(13)}$$

The brackets $[c_j,c_k]$ are the Lagrangian brackets.

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General Formula for a Lagrangian Bracket

Previously, a general set of orbital elements has been used. Before proceeding to evaluate the Lagrangian brackets of equations (11b), it is convenient to choose a specific set of independent orbital elements so as not to complicate the analysis. The set chosen is not significant; for, as will be shown in the RESULTS section, it is relatively simple to substitute any elements that may be desired. If there are chosen as the set of orbital elements the semimajor axis a, eccentricity e, time of pericenter passage t_p, argument of pericenter ω, orbit inclination I, and longitude of the ascending node Ω, the expression for the Lagrangian bracket is

$$[s,q] = \frac{\left(-t_p, -\frac{\mu}{2a}\right)}{(s,q)} + \frac{\left(\omega, \sqrt{\mu a(1 - e^2)}\right)}{(s,q)} + \frac{\left(\Omega, \sqrt{\mu a(1 - e^2)}\cos I\right)}{(s,q)} \tag{14}$$

where s and q are any of the orbital elements. The right-hand side is expressed in Jacobian notation. Note that $[s,s] = 0$ and $[s,q] = -[q,s]$ from equation (12). The derivation of equation (14) is given in appendix B. It results from geometric relations existing among the instantaneous values of the orbital elements of any orbit. It will be used to evaluate the Lagrangian brackets of equations (11b).

Evaluation of Lagrangian Brackets

A Lagrangian bracket appears in equations (11b) for each of the 36 combinations of the six chosen orbital elements. Evaluation from equation (14) shows six of the twelve nonzero brackets to be

$$[a,t_p] = -\frac{\partial(-t_p)}{\partial t_p}\frac{\partial(-\mu/2a)}{\partial a} = \frac{\mu}{2a^2} \tag{15a}$$

$$[a,\omega] = -\frac{\partial\omega}{\partial\omega}\frac{\partial\sqrt{\mu a(1 - e^2)}}{\partial a} = -\frac{1}{2}\sqrt{\frac{\mu}{a}}\sqrt{1 - e^2} \tag{15b}$$

$$[a,\Omega] = -\frac{\partial\Omega}{\partial\Omega}\frac{\partial\sqrt{\mu a(1 - e^2)}\cos I}{\partial a} = -\frac{1}{2}\cos I\sqrt{\frac{\mu}{a}}\sqrt{1 - e^2} \tag{15c}$$

$$[e,\omega] = -\frac{\partial\omega}{\partial\omega}\frac{\partial\sqrt{\mu a(1 - e^2)}}{\partial e} = \frac{e\sqrt{\mu a}}{\sqrt{1 - e^2}} \tag{15d}$$

$$[e,\Omega] = -\frac{\partial\Omega}{\partial\Omega}\,\frac{\partial\sqrt{\mu a(1 - e^2)}\cos I}{\partial e} = \frac{e\sqrt{\mu a}}{\sqrt{1 - e^2}}\cos I \tag{15e}$$

$$[I,\Omega] = -\frac{\partial\Omega}{\partial\Omega}\,\frac{\partial\sqrt{\mu a(1 - e^2)}\cos I}{\partial I} = \sqrt{\mu a(1 - e^2)}\sin I \tag{15f}$$

By observing the property of the brackets that $[s,q] = -[q,s]$, the values of the remaining six nonzero brackets are apparent from equations (15). Omitting all zero brackets, equations (11b) become

$$[a,t_p]\dot{t}_p + [a,\omega]\dot{\omega} + [a,\Omega]\dot{\Omega} = D_a \tag{16a}$$

$$[e,\omega]\dot{\omega} + [e,\Omega]\dot{\Omega} = D_e \tag{16b}$$

$$[I,\Omega]\dot{\Omega} = D_I \tag{16c}$$

$$-[a,t_p]\dot{a} = D_{t_p} \tag{16d}$$

$$-[a,\omega]\dot{a} - [e,\omega]\dot{e} = D_\omega \tag{16e}$$

$$-[a,\Omega]\dot{a} - [e,\Omega]\dot{e} - [I,\Omega]\dot{I} = D_\Omega \tag{16f}$$

Derivatives of Elements in Terms of Disturbing

Functions of Elements

Introducing the values obtained for the brackets in equations (15) into equations (16) and solving the system for the derivatives give the following form for the Lagrangian equations:

$$\dot{a} = -\frac{2a^2}{\mu} D_{t_p} \tag{17a}$$

$$\dot{e} = -\frac{a(1 - e^2)}{\mu e} D_{t_p} - \frac{1}{e}\sqrt{\frac{1 - e^2}{\mu a}}\, D_\omega \tag{17b}$$

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$$\dot{I} = \frac{1}{\sqrt{\mu a(1 - e^2)}} \left(D_\omega \cot I - \frac{D_\Omega}{\sin I}\right) \tag{17c}$$

$$\dot{t}_p = \frac{a(1 - e^2)}{\mu e} D_e + \frac{2a^2}{\mu} D_a \tag{17d}$$

$$\dot{\omega} = \frac{1}{e} \sqrt{\frac{1 - e^2}{\mu a}} D_e - \frac{\cot I}{\sqrt{\mu a(1 - e^2)}} D_I \tag{17e}$$

$$\dot{\Omega} = \frac{D_I}{\sin I \sqrt{\mu a(1 - e^2)}} \tag{17f}$$

Disturbing Functions of Elements in Terms of Components of Disturbing Acceleration

Prior to the integration of equations (17) it is desirable to express the disturbing functions D_a, D_e, D_{t_p}, D_ω, D_Ω, and D_I in more convenient form.

If s is any of the elements a, e, t_p, ω, Ω, or I, any of equations (13) may be written as

$$D_s = A_x \frac{\partial x}{\partial s} + A_y \frac{\partial y}{\partial s} + A_z \frac{\partial z}{\partial s} \tag{13a}$$

It is necessary only to evaluate $\partial x/\partial s$, $x \rightarrow y,z$ for each element in order to obtain useful forms of equations (17) that may be integrated (either formally or numerically). However, the equations will reduce to a more convenient form if the Cartesian disturbing acceleration components A_x, A_y, A_z are resolved into a new orthogonal set (fig. 2) as follows: (1) a component normal to the orbital plane W, positive when A_z is positive; (2) a component normal to the radius and in the orbital plane C, positive when making an angle of less than $\pi/2$ with the direction of motion; (3) a component along the radius R, positive when pointing outward from the origin.

The analysis in appendix C determines the disturbing functions D_a, D_e, D_{t_p}, D_ω, D_Ω, and D_I in terms W, C, and R as shown in equations (18).

$$D_a = \left[\frac{1 - e^2}{1 + e \cos v} - \frac{3}{2}\sqrt{\frac{\mu}{a^3(1 - e^2)}}\,(t - t_p)e \sin v\right]R - \left[\frac{3}{2}\sqrt{\frac{\mu}{a^3(1 - e^2)}}\,(t - t_p)(1 + e \cos v)\right]C \tag{18a}$$

$$D_e = (-a \cos v)R + a\left(\frac{2 + e \cos v}{1 + e \cos v}\right)(\sin v)C \tag{18b}$$

$$D_{t_p} = -\sqrt{\frac{\mu}{a(1 - e^2)}}\left[(e \sin v)R + (1 + e \cos v)C\right] \tag{18c}$$

$$D_\omega = Cr \tag{18d}$$

$$D_I = Wr \sin u \tag{18e}$$

$$D_\Omega = Cr \cos I - Wr \cos u \sin I \tag{18f}$$

where v is the true anomaly and $u = v + \omega$.

RESULTS

Introduction of equations (18) into equations (17) yields

$$\dot{a} = \frac{2a}{1 - e^2}\sqrt{\frac{p}{\mu}}\left[e(\sin v)R + \frac{p}{r}\,C\right] \tag{19a}$$

$$\dot{e} = \sqrt{\frac{p}{\mu}}\left\{(\sin v)R + \frac{1}{e}\left[\frac{p}{r} - \frac{r(1 - e^2)}{p}\right]C\right\} \tag{19b}$$

$$\dot{t}_p = \frac{a}{\mu}\left\{\left[2r - \frac{p}{e}\cos v - 3e\sqrt{\frac{\mu}{p}}\,(t - t_p)\sin v\right]R + \left[\frac{\sin v}{e}(p + r) - \frac{3}{r}\sqrt{\mu p}\,(t - t_p)\right]C\right\} \tag{19c}$$

$$\dot{\omega} = \sqrt{\frac{p}{\mu}}\left[-\frac{\cos v}{e}R + \frac{\sin v}{e}\left(1 + \frac{r}{p}\right)C - \left(\frac{r}{p}\sin u \cot I\right)W\right] \tag{19d}$$

$$\dot{\Omega} = \frac{r \sin u}{\sqrt{\mu p}\,\sin I}\,W \tag{19e}$$

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$$\dot{I} = \frac{r \cos u}{\sqrt{\mu p}} W \tag{19f}$$

where p is the semilatus rectum. These results, together with other forms derived from them, are given in table II.

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Alternate Components of Disturbing Acceleration

Components of the disturbing acceleration in the orbital plane may be alternately taken tangent to the path T and normal to the path N; T is positive in the direction of motion, and N is positive when directed toward the interior of the orbit. Substitution in terms of T and N for C and R by introducing expressions from table III gives the following changes in the derivatives of equations (19):

$$\dot{a} = 2a^2 \frac{V}{\mu} T \tag{20a}$$

$$\dot{e} = \frac{1}{V}\left\{2(e + \cos v)T - \left[\frac{r(1 - e^2)\sin v}{p}\right] N\right\} \tag{20b}$$

$$\dot{\omega} = \frac{1}{V}\left[\frac{2 \sin v}{e} T + \frac{r}{pe} (2e + \cos v + e^2 \cos v)N - \left(\frac{rV}{\sqrt{\mu p}} \sin u \cot I\right) W\right] \tag{20c}$$

$$\dot{t}_p = \frac{1}{V} \left\{\frac{1}{1 - e^2}\left[2 \frac{r}{e} \sqrt{\frac{p}{\mu}} (e^2 + e \cos v + 1)\sin v - 3(t - t_p)(1 + e^2 + 2e \cos v)\right] T + \left(\frac{r}{e} \sqrt{\frac{p}{\mu}} \cos v\right)N\right\} \tag{20d}$$

where V is the velocity.

Elimination of $(t - t_p)$

The quantity $(t - t_p)$ may be eliminated from equation (20d) by introducing table I equations (I-96) and (I-97) for the cases $e < 1$ and $e > 1$, respectively. The results for the two cases are

$$\dot{t}_p = \frac{a}{V} \sqrt{\frac{p}{\mu}} \left(\frac{1}{1 - e^2}\left\{\left[\frac{2}{e} + 3e + \frac{e^2(e + \cos v)}{1 + e \cos v}\right] \sin v - \frac{3(1 + 2e \cos v + e^2)}{\sqrt{1 - e^2}} E\right\}T + \left(\frac{r}{ae} \cos v\right) N\right), \qquad 0 < e < 1 \tag{21}$$

where E is the eccentric anomaly, and

$$\dot{t}_p = \frac{a}{V}\sqrt{\frac{p}{\mu}}\left(\frac{1}{1 - e^2}\left\{\left[\frac{2}{e} + 3e + \frac{e^2(e + \cos v)}{1 + e \cos v}\right] \sin v\right.\right.$$

$$\left.\left. - \frac{3(1 + 2e \cos v + e^2)}{\sqrt{e^2 - 1}} F\right\} T + \left(\frac{r}{ae} \cos v\right) N\right), \qquad e > 1 \qquad (22)$$

where $F = -iE$ in the hyperbolic orbit corresponds to E in the elliptic orbit. Thus, F is imaginary when an elliptic orbit exists, and E is imaginary in the case of the hyperbolic orbit. The result for $e > 1$ is identical to the result for $e < 1$, but with E replaced by iF and $\sqrt{1 - e^2}$ replaced by $i\sqrt{e^2 - 1}$.

The value for $\dot{t}_p$ when $e = 1$ is not directly evident as the quantity $(1 - e^2) \rightarrow 0$ when $e \rightarrow 1$. Equation (19c) for $\dot{t}_p$ may be written as

$$\dot{t}_p = \frac{p^2}{\mu(1 - e^2)}\left\{\left[\frac{2}{1 + e \cos v} - \frac{\cos v}{e} - 3e \sin v \sqrt{\frac{\mu}{p^3}}\,(t - t_p)\right] R\right.$$

$$\left. + \left[\frac{\sin v}{e}\,\frac{2 + e \cos v}{1 + e \cos v} - 3(1 + e \cos v)\sqrt{\frac{\mu}{p^3}}\,(t - t_p)\right] C\right\} \qquad (23)$$

Eliminating $\sqrt{\frac{\mu}{p^3}}\,(t - t_p)$ from equation (23) with table I equation (I-95) yields

$$\dot{t}_p = \frac{p^2}{\mu(1 - e^2)}\left\{\left[\frac{2}{1 + e \cos v} - \frac{\cos v}{e} + \frac{3e^2 \sin^2 v}{(1 - e^2)(1 + e \cos v)}\right.\right.$$

$$\left. - \frac{3e \sin v}{1 - e^2}\int_0^v \frac{dv}{1 + e \cos v}\right] R + \left[\frac{\sin v}{e}\left(\frac{2 + e \cos v}{1 + e \cos v}\right)\right.$$

$$\left.\left. + \frac{3e \sin v}{1 - e^2} - \frac{3(1 + e \cos v)}{1 - e^2}\int_0^v \frac{dv}{1 + e \cos v}\right] C\right\} \qquad (24)$$

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Using the relation

$$\int_0^v \frac{\cos v\, dv}{(1 + e \cos v)^3} = \frac{1}{2(1 - e^2)} \left[\frac{\sin v}{(1 + e \cos v)^2} + \frac{(1 + 2e^2)\sin v}{(1 - e^2)(1 + e \cos v)} - \frac{3e}{1 - e^2} \int_0^v \frac{dv}{1 + e \cos v} \right] \tag{25}$$

to eliminate $\int_0^v \frac{dv}{1 + e \cos v}$ from equation (24) yields

$$\dot{t}_p = \frac{p^2}{\mu} \left\{ \left[2 \sin v \int_0^v \frac{\cos v\, dv}{(1 + e \cos v)^3} - \frac{\cos v}{e(1 + e \cos v)^2} \right] R + \left[\frac{2}{e} (1 + e \cos v) \int_0^v \frac{\cos v\, dv}{(1 + e \cos v)^3} \right] C \right\} \tag{26}$$

Equation (26) is now defined at $e = 1$, since

$$\int_0^v \frac{\cos v\, dv}{(1 + \cos v)^3} = \frac{\sin v}{5} \; \frac{1 + 3 \cos v + \cos^2 v}{(1 + \cos v)^3}$$

Hence, the equation for $\dot{t}_p$ on a parabola is

$$\dot{t}_p = \frac{p^2}{5\mu(1 + \cos v)^2} \left\{ \left[2(1 + 3 \cos v + \cos^2 v)\sin v \right] C + \left[2 - \cos v - 4 \cos^2 v - 2 \cos^3 v \right] R \right\} \tag{27}$$

Derivatives of Alternate Orbital Parameters

The element "p" may be used in place of the element "a" as a variable of integration. Similarly, the pericenter radius r_p may replace a or p. As p, a, and r_p are similar elements, the question of which to select is determined by numerical and convenience considerations.

Semilatus rectum. - Taking the derivative of $p = a(1 - e^2)$ yields $\dot{p} = \dot{a}(1 - e^2) - 2ea\dot{e}$. Introducing expressions for $\dot{a}$ and $\dot{e}$ from equations (19a) and (19b) yields

$$\dot{p} = 2r \sqrt{\frac{p}{\mu}}\, C \tag{28}$$

or, in terms of T and N,

$$\dot{p} = \frac{2p}{V}\left[T + \frac{r}{p}\, e(\sin v)N\right] \tag{29}$$

Pericenter radius. - The equation for radius of pericenter is $r_p = p/(1 + e)$, and its derivative is

$$\dot{r}_p = \frac{1}{1 + e}\,(\dot{p} - r_p\dot{e}) \tag{30}$$

Introducing expressions for $\dot{p}$ and $\dot{e}$ from equations (29) and (20b) into equation (30) and simplifying yield

$$\dot{r}_p = \frac{1}{V}\left[2r_p\,\frac{1 - \cos v}{1 + e}\,T + (r \sin v)N\right] \tag{31}$$

Mean anomaly. - Also, it may be desirable to use the mean anomaly M in place of the element t_p. Here, the distinction is significant in that M is not an element and varies even on a two-body orbit. From the familiar expressions $M = n(t - t_p)$ and $n = \sqrt{\mu/a^3}$ in table I, the derivatives become

$$\dot{M} = \dot{n}(t - t_p) + n - n\dot{t}_p \tag{32}$$

and

$$\dot{n} = -\frac{3}{2}\sqrt{\frac{\mu}{a^5}}\,\dot{a} = -3\,\frac{V}{\sqrt{\mu a}}\,T \tag{33}$$

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where equation (20a) was used for $\dot{a}$ in equation (33). Now,

$$-n\dot{t}_p = -\sqrt{\frac{\mu}{a^3}}\,\dot{t}_p$$

$$= 3(t - t_p)\frac{V}{\sqrt{\mu a}}T - \frac{1}{V}\sqrt{\frac{p}{a^3}}\left[\frac{2}{1 - e^2}r(\sin v)\left(e + \frac{p}{re}\right)T + \frac{r\cos v}{e}N\right] \tag{34}$$

when $\dot{t}_p$ is taken from equation (20d).

Introducing equations (33) and (34) into equation (32) yields

$$\dot{M} = n - \frac{\sqrt{1 - e^2}}{V}\left[2(\sin v)\left(\frac{re}{p} + \frac{1}{e}\right)T + \frac{r\cos v}{ae}N\right] \tag{35}$$

It should be noted that M and $\dot{M}$ are both zero when $e = 1$ and that $\dot{M}$ reduces to its two-body value when the in-plane components of the disturbing acceleration are zero.

True anomaly. - The orbital parameter v, true anomaly, may be used as an alternate to either the element t_p or the parameter M. From table I equations (I-47), (I-18), and (I-24), Kepler's equation may be written as

$$M = \tan^{-1}\frac{\sqrt{1 - e^2}\sin v}{e + \cos v} - e\frac{\sqrt{1 - e^2}\sin v}{1 + e\cos v} \tag{36}$$

Taking derivatives in the disturbed orbit yields

$$\dot{M} = \frac{\sqrt{1 - e^2}}{(1 + e\cos v)^2}\left[\dot{v}(1 - e^2) - \dot{e}(\sin v)(2 + e\cos v)\right] \tag{37}$$

Solving equation (37) for $\dot{v}$ and introducing equations (35) and (20b) to eliminate $\dot{M}$ and $\dot{e}$, respectively, yield the result

$$\dot{v} = \frac{\sqrt{\mu p}}{r^2} - \left(\frac{2\sin v}{Ve}\right)T - \frac{2e + e^2\cos v + \cos v}{Ve(1 + e\cos v)}N \tag{38}$$

As in the case of $\dot{M}$ (eq. (35)), it will be noted that equation (38) reduces to the two-body derivative expression when the in-plane components of the perturbative acceleration are zero.

Table IV is a qualitative summary showing how the components of the disturbing acceleration affect the derivatives of the various orbital elements and parameters.

Orbital Element Relations for a Circular Orbit

In a circular orbit the location of the pericenter is undefined. Consequently, the elements ω and t_p and the true and mean anomalies v and M, which are related to the location of the pericenter, are undefined. Thus, relations involving ω, t_p, v, and M take on an indeterminate form. However, any perturbation having a component in the plane of the orbit will immediately establish the limiting values of ω, t_p, v, and M in the circular orbit. The circular orbit expressions are derived for the T, N system of resolution of the in-plane disturbing acceleration components. The derivatives are also given in terms of the C, R system in table II.

True anomaly. - Taking the table I equation (I-117) for $\cos v$,

$$\cos v = \frac{(p/r) - 1}{e} = \frac{0}{0}, \qquad e = 0$$

and applying L'Hospital's rule give

$$\cos v = \frac{(r\dot{p} - p\dot{r})/r^2}{\dot{e}}, \qquad e = 0 \tag{39}$$

Substituting for $\dot{p}$ from equation (29), for $\dot{e}$ from equation (20b), and for $\dot{r}$ from table I equation (I-80) [eq. (I-80) is also valid for disturbed orbits] into equation (39) yields in the limit as e approaches zero

$$\cos v = \frac{2T}{2T \cos v - N \sin v}, \qquad e = 0 \tag{40}$$

Equation (40) is valid if $e = 0$ and $\dot{e} \neq 0$. The latter condition requires that an in-plane component of the perturbing acceleration exists. Solving for $\sin v$ and $\cos v$ from equation (40) yields

$$\sin v = \mp \frac{N}{\sqrt{4T^2 + N^2}}, \qquad e = 0 \tag{41a}$$

$$\cos v = \pm \frac{2T}{\sqrt{4T^2 + N^2}}, \qquad e = 0 \tag{41b}$$

where the upper sign gives the limiting value of the true anomaly for leaving a circular orbit and the lower sign gives the limiting value when entering a circular orbit. It can be seen from equations (41) that, if a perturbing force is arranged to force an initially elliptical orbit through circular, the value of true anomaly will make a step change of π radians.

Argument of pericenter. - The argument of pericenter is determined from table I equation (I-176):

$$\omega = u - v$$

Time of pericenter passage. - In a circular orbit the time from "pericenter," $t - t_p$, equals the arc length to "pericenter" divided by the velocity; that is,

$$t - t_p = \frac{r}{V} v, \qquad e = 0 \tag{42}$$

Derivatives of Orbital Parameters in a Circular Orbit

Semilatus rectum and semimajor axis. - Equations (29) and (20a) for $\dot{p}$ and $\dot{a}$ reduce directly to the same expression when $e = 0$:

$$\dot{p} = \dot{a} = 2 \frac{r}{V} T, \qquad e = 0 \tag{43}$$

Eccentricity and radius of pericenter. - Using the limiting values for $\sin v$ and $\cos v$ from equations (41) in equations (20b) and (31) for $\dot{e}$ and $\dot{r}_p$ yields the following:

$$\dot{e} = \pm \frac{1}{V} \sqrt{4T^2 + N^2}, \qquad e = 0 \tag{44}$$

$$\dot{r}_p = \frac{r}{V} \left(2T \mp \sqrt{4T^2 + N^2}\right), \qquad e = 0 \tag{45}$$

where again the upper signs give the limiting values of $\dot{e}$ and $\dot{r}_p$ when leaving a circular orbit and the lower signs are for entering a circular orbit.

Argument of pericenter. - Equation (20c) may be rewritten to give

$$\dot{\omega} = \frac{1}{V}\left[\frac{1}{e}\left(2T \sin v + \frac{r}{p} N \cos v\right) + \frac{r}{p} (2 + e \cos v)N - \left(\frac{rV}{\sqrt{\mu p}} \sin u \cot I\right)W\right] \tag{46}$$

Substituting the following relations from table I

$$\sin v = \sqrt{\frac{p}{\mu}}\frac{\dot{r}}{e} \tag{47}$$

$$\cos v = \frac{1}{e}\left(\frac{p}{r} - 1\right) \tag{48}$$

gives the following result in the first term of equation (46):

$$\frac{1}{e}\left(2T \sin v + \frac{r}{p} N \cos v\right) = \frac{1}{e^2}\left[2T \sqrt{\frac{p}{\mu}}\dot{r} + N\left(1 - \frac{r}{p}\right)\right]$$

Applying L'Hospital's rule and using equations (20b), (29), (43), (47), (48) and equations from table I and noting that $\ddot{r} = -N$ in the disturbed orbit when $e = 0$, it is found that

$$\lim_{e \to 0} \frac{2T \sqrt{\frac{p}{\mu}}\dot{r} + N\left(1 - \frac{r}{p}\right)}{e^2} = \frac{V^2}{2r} - N\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right) + V\left(\frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}\right)$$

Thus, $\dot{\omega}$ reduces to the following for a circular orbit:

$$\dot{\omega} = \frac{1}{V}\left[\frac{V^2}{2r} + N \frac{2T^2 + N^2}{4T^2 + N^2} + V \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2} - (\sin u \cot I)W\right], \qquad e = 0 \tag{49}$$

Time of pericenter passage. - Equation (21) may be rewritten to give

$$\dot{t}_p = \frac{p}{V}\sqrt{\frac{p}{\mu}}\left\{\frac{1}{e}\left[\frac{2T \sin v}{1 - e^2} + \frac{r}{p} N \cos v\right] + \frac{eT \sin v}{1 - e^2}\left[3 - e(e + \cos v)\frac{r}{p}\right] - \frac{3E(1 + 2e \cos v + e^2)}{\sqrt{(1 - e^2)^5}} T\right\} \tag{50}$$

Again, the limit of the first term of $\dot{t}_p$ is

$$\lim_{e \to 0} \frac{\left(\frac{2T \sin v}{1 - e^2} + \frac{r}{p} N \cos v\right)}{e} = \frac{V^2}{2r} - N\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right) + V\left(\frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}\right)$$

and $\dot{t}_p$ for a circular orbit reduces to

$$\dot{t}_p = \frac{1}{2} + \frac{r}{V}\left(\frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}\right) - \frac{r}{V^2}\left(\frac{6T^2 + N^2}{4T^2 + N^2} N + 3vT\right), \qquad e = 0 \tag{51}$$

Mean anomaly. - Equation (35) may be written as

$$\dot{M} = n - \frac{\sqrt{1 - e^2}}{V}\left[\frac{2(\sin v)T + \frac{r}{p}(1 - e^2)N \cos v}{e} + 2\left(\frac{re}{p} \sin v\right)T\right] \tag{52}$$

The limit of the indeterminate term in equation (52) is

$$\underset{e \to 0}{\text{Limit}} \frac{2T \sin v + \frac{r}{p}(1 - e^2)N \cos v}{e} = \frac{V^2}{2r} - N\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right) \mp \frac{2TN}{\sqrt{4T^2 + N^2}} + V\left(\frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}\right)$$

and $\dot{M}$ for a circular orbit becomes

$$\dot{M} = n - \frac{V}{2r} + \frac{N}{V}\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right) \pm \frac{2TN}{V\sqrt{4T^2 + N^2}} - \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2} \tag{53}$$

where the upper sign is for leaving a circular orbit and the lower sign is for entering a circular orbit.

True anomaly. - Equation (38) may be written as

$$\dot{v} = -\frac{1}{V}\left[\frac{1}{e}\left(2T \sin v + \frac{r}{p} N \cos v\right)\right] - \frac{r}{Vp}(2 + e \cos v)N + \frac{\sqrt{\mu p}}{r^2}$$

which yields in the limit as e approaches zero

$$\dot{v} = \frac{V}{2r} - \frac{N}{V}\left(\frac{2T^2 + N^2}{4T^2 + N^2}\right) - \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}, \qquad e = 0 \tag{54}$$

Application of derivatives for circular orbits. - If the disturbing force is applied in a constant direction relative to the velocity direction, it may be noted that

$$T\dot{N} - N\dot{T} = 0$$

even if the magnitude of the disturbing acceleration is varying. Thus, if the direction of the disturbing acceleration is fixed relative to the velocity, all terms involving $\dot{N}$ and $\dot{T}$ will vanish in equations (49) to (54). Similarly if the direction of the disturbing acceleration is fixed relative to the radial direction, all terms involving $\dot{C}$ and $\dot{R}$ will vanish from the equations in table II.

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The equations for circular orbits given in the preceding sections and in table II are completely valid only for circular orbits. However, certain of the equations will be found to be sufficiently accurate for near-circular orbits and can be used as the basis of approximate equations.

CONCLUDING REMARKS

All the results derived herein for the perturbation derivatives of the various orbital elements and parameters are listed in table II. Table III contains expressions for the orthogonal system of components C, R, and W in terms of the Cartesian components of the disturbing acceleration A_x, A_y, A_z in the OX, OY, OZ system. It also contains equations interrelating the C, R and T, N systems of the in-plane components of the disturbing acceleration. Table I is a collection of various forms of the two-body equations that also apply to the osculating orbit.

As illustrated for $\dot{p}$, $\dot{r}_p$, $\dot{M}$, and $\dot{v}$, perturbation derivatives of other alternate elements and parameters may be derived from the expressions in tables I, II, and III. Other integration variables that may be useful for the solution of problems in orbital mechanics are suggested in references 2, 3, and 4.

Selection of the best set of orbital elements or parameters for a particular type of special perturbation problem in orbital mechanics depends on the nature of the problem. However, it is expected that examination of the derivatives given herein will help to indicate which parameters should be used for a specific problem.

Lewis Research Center
National Aeronautics and Space Administration
Cleveland, Ohio, August 30, 1961

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APPENDIX A

SYMBOLS

The following symbols are used in this report:

A_x, A_y, A_z	component of disturbing acceleration
a	semimajor axis of conic section, negative in hyperbolic case
b	semiminor axis of conic section
C	perturbative acceleration in circumferential direction
c	constant of integration
D_s	disturbing acceleration function for element s, $A_x \frac{\partial x}{\partial s} + A_y \frac{\partial y}{\partial s} + A_z \frac{\partial z}{\partial s}$
E	eccentric anomaly
Eg	energy per unit mass
e	eccentricity
F	used in hyperbolic orbits to correspond to eccentric anomaly in elliptic orbits, $F = -iE$
f	component term of disturbing acceleration includes forces due to all except gravitating bodies not located at problem origin
G	$a^{3/2}(E - e \sin E)$
H	$\frac{1}{2} \mu^{1/2} a^{1/2}(3E + e \sin E)$
h	angular momentum per unit mass equals twice the rate of description of area in orbital plane
I	orbital plane inclination
i	unit imaginary number, $\sqrt{-1}$
J	$\dot{X} \frac{\partial X}{\partial s} + \dot{Y} \frac{\partial Y}{\partial s}$

k^2	gravitational constant
M	mean anomaly
M_i	gravitating body mass
M_o	gravitating body mass at problem origin
m	object mass
N	perturbative acceleration in orbital plane in direction normal to velocity, positive when directed toward interior of orbit
n	mean angular orbital motion of object, $2\pi/P$
P	orbital period
p	semilatus rectum, $a(1 - e^2)$
Q,S	denote functions defined for convenience
q,s	any pair of orbital elements
R	perturbative acceleration in radial direction, positive outward
Rg	range on surface of sphere intersected by an elliptical orbit
r	radius from origin to object
r_i	radius from origin to disturbing body
r_p	pericenter radius, the minimum distance from central body to orbit
r_s	radius of sphere
T	perturbative acceleration in direction of velocity
t	time
t_p	time of pericenter
U,U_1	complex variables, $x + iy$, $x_1 + iy_1$
u	argument of latitude, angle measured from ascending node to object radius in the direction of motion, $u = v + \omega$

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V	object velocity or complex variable, $x - iy$
V_1	complex variable, $x_1 - iy_1$
v	true anomaly, angle measured from pericenter to object radius in the direction of motion
W	perturbative acceleration normal to orbital plane, positive in the direction of axis OZ
X,Y	denote object coordinates in the OX_3, OY_3 axis system where OX_3 coincides with pericenter radius and OY_3 lies in orbital plane (see fig. 2)
x,y,z	denote object coordinates in the OX, OY, OZ Cartesian system (see fig. 1)
$\alpha_1,\beta_1,\gamma_1$	direction cosines of the OX_3 axis referred to the OX, OY, and OZ axes, respectively
$\alpha_2,\beta_2,\gamma_2$	direction cosines of the OY_3 axes referred to the OX, OY, and OZ axes, respectively
Δ	distance from object to any perturbing gravitating body
ϵ	natural logarithm base
μ	gravitational constant equals acceleration of object at unit distance from M_o due to M_o, $\mu = k^2(M_o + m)$
ν	used in hyperbolic orbits to correspond to n in elliptic orbits, $\nu = -in$
ψ	path angle, angle between circumferential and velocity directions, positive clockwise
ω	argument of pericenter, angle measured from ascending node to pericenter radius in direction of motion
Ω	longitude of ascending node

Subscripts:

a	apocenter conditions in table I; elsewhere, disturbing function of semimajor axis
e	disturbing function of e

I	disturbing function of I
$i=1,2,\cdots,n$	disturbing gravitating body number
$j=1,2,\cdots,6$	indicates j^{th} equation of a set
$k=1,2,\cdots,6$	indicates k^{th} orbital element
p	pericenter conditions
s	conditions of spherical surface
t_p	disturbing function of t_p
$x \rightarrow y,z$	indicates extension of equations in x to a system including y and z
x,y,z	components taken about or along an axis
ω	disturbing function of ω
1,2,3	coordinate system and coordinates in them
Ω	disturbing function of Ω

Superscript:

$\cdot$	indicates derivative with respect to time

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APPENDIX B

WHITTAKER'S DERIVATION OF THE GENERAL FORMULA FOR A LAGRANGIAN BRACKET

The following development is applicable to the equations of any osculating orbit. It is similar to that given in article 5-16 of Smart (ref. 1).

Let s and q be any two of the six orbital elements and x, y, z be the object's coordinates; then equation (12) may be written as a sum of Jacobians,

$$[s,q] = \frac{(x,\dot{x})}{(s,q)} + \frac{(y,\dot{y})}{(s,q)} + \frac{(z,\dot{z})}{(s,q)} \tag{B1}$$

where, by definition,

$$\frac{(x,\dot{x})}{(s,q)} = \frac{\partial x}{\partial s}\frac{\partial \dot{x}}{\partial q} - \frac{\partial x}{\partial q}\frac{\partial \dot{x}}{\partial s}$$

and so forth, for the other Jacobians. The objective is to transform equation (B1) into an expression of the same form but involving only explicit functions of the orbital elements. Define

$$\left.\begin{aligned} S &= \dot{x}\frac{\partial x}{\partial s} + \dot{y}\frac{\partial y}{\partial s} + \dot{z}\frac{\partial z}{\partial s} \\ &\text{and} \\ Q &= \dot{x}\frac{\partial x}{\partial q} + \dot{y}\frac{\partial y}{\partial q} + \dot{z}\frac{\partial z}{\partial q} \end{aligned}\right\} \tag{B2}$$

Then,

$$[s,q] = \frac{\partial S}{\partial q} - \frac{\partial Q}{\partial s} \tag{B3}$$

Let the OX, OY, OZ axis system in figure 1 be fixed to some reference astronomical line and plane such as the mean equinox and equator of 1950.0. The orbit plane is projected on the celestial sphere. The equatorial longitude of the ascending equatorial node referred to the vernal equinox is then denoted by Ω, and the inclination of the orbital plane to the equator is I. Rotate OX and OY about OZ through the angle Ω to obtain the axis system OX_1, OY_1, OZ_1. Coordinates of the object in the OX, OY, OZ system (x,y,z) are given in terms of those in the OX_1, OY_1, OZ_1 system (x_1,y_1,z_1) by

$$\left.\begin{aligned} x &= x_1 \cos \Omega - y_1 \sin \Omega \\ y &= x_1 \sin \Omega + y_1 \cos \Omega \\ z &= z_1 \end{aligned}\right\} \tag{B4}$$

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It is convenient to introduce complex variables by the following definitions:

$$\left.\begin{aligned} &U = x + iy, \qquad V = x - iy \\ &\text{and} \\ &U_1 = x_1 + iy_1, \qquad V_1 = x_1 - iy_1 \end{aligned}\right\} \tag{B5}$$

where $i = \sqrt{-1}$. Then, by using equations (B4) with the definitions in equations (B5), it may be shown that

$$U = U_1(\cos \Omega + i \sin \Omega) \tag{B6a}$$

$$V = V_1(\cos \Omega - i \sin \Omega) \tag{B6b}$$

or

$$U = U_1 \epsilon^{i\Omega} \tag{B7a}$$

$$V = V_1 \epsilon^{-i\Omega} \tag{B7b}$$

The time derivative of equation (B7a) and the partial derivative of equation (B7b) with respect to s are, respectively,

$$\dot{U} = \dot{U}_1 \epsilon^{i\Omega} \qquad \text{and} \qquad \frac{\partial V}{\partial s} = \epsilon^{-i\Omega}\left(\frac{\partial V_1}{\partial s} - iV_1 \frac{\partial \Omega}{\partial s}\right) \tag{B8}$$

Multiply equations (B8) together to obtain

$$\dot{U} \frac{\partial V}{\partial s} = \dot{U}_1 \frac{\partial V_1}{\partial s} - i\dot{U}_1 V_1 \frac{\partial \Omega}{\partial s} \tag{B9}$$

Taking the real parts of equation (B9) yields

$$\dot{x}\,\frac{\partial x}{\partial s} + \dot{y}\,\frac{\partial y}{\partial s} = \dot{x}_1\,\frac{\partial x_1}{\partial s} + \dot{y}_1\,\frac{\partial y_1}{\partial s} + (x_1\dot{y}_1 - \dot{x}_1 y_1)\,\frac{\partial \Omega}{\partial s} \tag{B10}$$

But it is seen from equations (I-33) and (I-44) of the two-body orbital relations in table I that $x_1\dot{y}_1 - \dot{x}_1 y_1 = h \cos I$, which is the projection on the $OX_1 - OY_1$ plane of twice the rate of description of area in the orbit plane. Hence, from equations (B2) and (B10)

$$S = \dot{x}_1\,\frac{\partial x_1}{\partial s} + \dot{y}_1\,\frac{\partial y_1}{\partial s} + \dot{z}_1\,\frac{\partial z_1}{\partial s} + h \cos I\,\frac{\partial \Omega}{\partial s} \tag{B11}$$

as $z = z_1$ from equation (B4). Now, rotate the $OY_1 - OZ_1$ plane about OX_1 through the angle I to obtain the axis system OX_2, OY_2, OZ_2 having object coordinates (x_2,y_2,z_2), so that the $OX_2 - OY_2$ plane coincides with the orbit plane. Then, by analogy with equation (B10),

$$\dot{y}_1\,\frac{\partial y_1}{\partial s} + \dot{z}_1\,\frac{\partial z_1}{\partial s} = \dot{y}_2\,\frac{\partial y_2}{\partial s} + \dot{z}_2\,\frac{\partial z_2}{\partial s} + (\dot{z}_2 y_2 - \dot{y}_2 z_2)\,\frac{\partial I}{\partial s} \tag{B12}$$

But, since $x_1 = x_2$ and $z_2 \equiv 0$, equations (B11) and (B12) may be used to yield

$$S = \dot{x}_2\,\frac{\partial x_2}{\partial s} + \dot{y}_2\,\frac{\partial y_2}{\partial s} + h \cos I\,\frac{\partial \Omega}{\partial s} \tag{B13}$$

Now rotate the $OX_2 - OY_2$ plane in the orbital plane through the angle ω to obtain the axis system OX_3, OY_3 having object coordinates (X,Y) so that OX_3 lies on the pericenter. Again, by analogy with equation (B10) it follows that

$$\dot{x}_2\,\frac{\partial x_2}{\partial s} + \dot{y}_2\,\frac{\partial y_2}{\partial s} = \dot{X}\,\frac{\partial X}{\partial s} + \dot{Y}\,\frac{\partial Y}{\partial s} + (X\dot{Y} - Y\dot{X})\,\frac{\partial \omega}{\partial s} \tag{B14}$$

But $X\dot{Y} - Y\dot{X} = h$; thus, use of equations (B13) and (B14) yields

$$S = \dot{X}\,\frac{\partial X}{\partial s} + \dot{Y}\,\frac{\partial Y}{\partial s} + h\,\frac{\partial \omega}{\partial s} + h \cos I\,\frac{\partial \Omega}{\partial s} \tag{B15}$$

Let $J = \dot{X} \frac{\partial X}{\partial s} + \dot{Y} \frac{\partial Y}{\partial s}$; then substitutions from table I equations (I-144), (I-151), (I-158), and (I-163) may be introduced to obtain

$$\frac{J}{\dot{E}} = a^2 \frac{\partial E}{\partial s} (1 - e^2 \cos^2 E) + a(\sin E)(1 - e \cos E)\left(e \frac{\partial a}{\partial s} + a \frac{\partial e}{\partial s}\right) \tag{B16}$$

where E is the eccentric anomaly.

The time derivative of Kepler's equation (eqs. (I-47) and (I-52)) is

$$\dot{E}(1 - e \cos E) = n = \mu^{1/2} a^{-(3/2)} \tag{B17}$$

where n is the mean angular motion in the orbit. Eliminate $\dot{E}$ in equation (B16) using equation (B17) to obtain

$$J = \mu^{1/2}\left[a^{1/2}(1 + e \cos E) \frac{\partial E}{\partial s} + a^{-(1/2)} e \sin E \frac{\partial a}{\partial s} + a^{1/2} \sin E \frac{\partial e}{\partial s}\right]$$

which can be written as

$$\begin{aligned} J = \mu^{1/2}\Bigg[&-\frac{1}{2} a^{1/2}(1 - e \cos E) \frac{\partial E}{\partial s} - \frac{3}{4} a^{-(1/2)}(E - e \sin E) \frac{\partial a}{\partial s} \\ &+ \frac{1}{2} a^{1/2}(3 + e \cos E) \frac{\partial E}{\partial s} + \frac{1}{4} a^{-(1/2)}(3E + e \sin E) \frac{\partial a}{\partial s} \\ &+ a^{1/2} \sin E \frac{\partial e}{\partial s}\Bigg] \end{aligned}$$

$$J = -\frac{\mu^{1/2}}{2a} \frac{\partial}{\partial s}\left[a^{3/2}(E - e \sin E)\right] + \frac{\mu^{1/2}}{2} \frac{\partial}{\partial s}\left[a^{1/2}(3E + e \sin E)\right]$$

$$J \equiv -\frac{\mu^{1/2}}{2a} \frac{\partial G}{\partial s} + \frac{\partial H}{\partial s} \tag{B18}$$

where $H \equiv \frac{1}{2} \mu^{1/2} a^{1/2}(3E + e \sin E)$ and

$G \equiv a^{3/2}(E - e \sin E) = na^{3/2}(t - t_p) = \mu^{1/2}(t - t_p)$. Thus,

$$\frac{\partial G}{\partial s} = \mu^{1/2} \frac{\partial(-t_p)}{\partial s} \tag{B19}$$

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Introduce equations (B18) and (B19) into equation (B15) to obtain

$$S = -\frac{\mu}{2a}\frac{\partial(-t_p)}{\partial s} + \frac{\partial H}{\partial s} + h\frac{\partial \omega}{\partial s} + h \cos I \frac{\partial \Omega}{\partial s} \tag{B20a}$$

A similar derivation will show that Q is given by

$$Q = -\frac{\mu}{2a}\frac{\partial(-t_p)}{\partial q} + \frac{\partial H}{\partial q} + h\frac{\partial \omega}{\partial q} + h \cos I \frac{\partial \Omega}{\partial q} \tag{B20b}$$

Using equations (B20), equation (B3) for Lagrange's brackets may now be written as

$$[s,q] = \frac{\partial}{\partial s}(-t_p)\frac{\partial}{\partial q}\left(-\frac{\mu}{2a}\right) - \frac{\partial}{\partial s}\left(-\frac{\mu}{2a}\right)\frac{\partial}{\partial q}(-t_p) + \frac{\partial \omega}{\partial s}\frac{\partial h}{\partial q} - \frac{\partial h}{\partial s}\frac{\partial \omega}{\partial q}$$

$$+ \frac{\partial \Omega}{\partial s}\frac{\partial(h \cos I)}{\partial q} - \frac{\partial(h \cos I)}{\partial s}\frac{\partial \Omega}{\partial q} \tag{B21}$$

Using Jacobian notation, the general expression for a Lagrangian bracket as obtained by rewriting equation (B21) is

$$[s,q] = \frac{\left(-t_p, -\frac{\mu}{2a}\right)}{(s,q)} + \frac{(\omega,h)}{(s,q)} + \frac{(\Omega,h \cos I)}{(s,q)} \tag{B22}$$

Substitution of $h = \sqrt{\mu a(1 - e^2)}$ into equation (B22) gives the expression from which the Lagrangian brackets will be evaluated for the chosen elements

$$[s,q] = \frac{\left(-t_p, -\frac{\mu}{2a}\right)}{(s,q)} + \frac{\left(\omega, \sqrt{\mu a(1 - e^2)}\right)}{(s,q)} + \frac{\left(\Omega, \sqrt{\mu a(1 - e^2)}\cos I\right)}{(s,q)} \tag{B23}$$

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APPENDIX C

DISTURBING FUNCTIONS OF THE ELEMENTS IN TERMS OF COMPONENTS OF THE DISTURBING ACCELERATION

By the application of spherical trigonometry it is seen that the components A_x, A_y, A_z of the disturbing acceleration referred to OX, OY, and OZ in figure 1 are given in terms of W, C, and R by

$$A_x = R(\alpha_1 \cos v + \alpha_2 \sin v) + C(\alpha_2 \cos v - \alpha_1 \sin v) + W \sin \Omega \sin I \tag{C1a}$$

$$A_y = R(\beta_1 \cos v + \beta_2 \sin v) + C(\beta_2 \cos v - \beta_1 \sin v) - W \cos \Omega \sin I \tag{C1b}$$

$$A_z = R(\gamma_1 \cos v + \gamma_2 \sin v) + C(\gamma_2 \cos v - \gamma_1 \sin v) + W \cos I \tag{C1c}$$

where

$$\left.\begin{aligned}
\alpha_1 &= \cos \omega \cos \Omega - \sin \omega \sin \Omega \cos I \\
\alpha_2 &= -\sin \omega \cos \Omega - \cos \omega \sin \Omega \cos I \\
\beta_1 &= \cos \omega \sin \Omega + \sin \omega \cos \Omega \cos I \\
\beta_2 &= -\sin \omega \sin \Omega + \cos \omega \cos \Omega \cos I \\
\gamma_1 &= \sin \omega \sin I \\
\gamma_2 &= \cos \omega \sin I
\end{aligned}\right\} \tag{C2}$$

Geometrically, α_1, β_1, γ_1 are the direction cosines of the pericenter radius, and α_2, β_2, γ_2 are the direction cosines of the circumferential direction at pericenter referred to OX, OY, OZ, respectively.

To obtain $\partial x/\partial s$, $x \to y,z$ where s is any of the orbital elements, the coordinates of the object must be expressed in terms of the orbital elements. The (x,y,z) coordinates in the system OX, OY, OZ may be expressed in terms of the orbital elements and true anomaly v as

$$x = \frac{a(1 - e^2)}{1 + e \cos v} (\alpha_1 \cos v + \alpha_2 \sin v) \qquad \text{(C3a)}$$

$$y = \frac{a(1 - e^2)}{1 + e \cos v} (\beta_1 \cos v + \beta_2 \sin v) \qquad \text{(C3b)}$$

$$z = \frac{a(1 - e^2)}{1 + e \cos v} (\gamma_1 \cos v + \gamma_2 \sin v) \qquad \text{(C3c)}$$

when equations (I-140), (I-147), (I-154), (I-72), and (I-104) from table I are used.

Determination of D_a, D_e, D_{t_p}

From equations (C2) it is seen that the derivatives of α, β, and γ with respect to the elements a, e, and t_p are zero. Equations (C3) are not explicit in terms of the chosen set of orbital elements, for $\sin v$ and $\cos v$ are seen to be functions of a, e, and t_p by the following form of Kepler's equation obtained from equations (I-18), (I-24), (I-45), (I-47), and (I-52) of table I:

$$(t - t_p)\mu^{1/2}a^{-3/2} = \tan^{-1}\left(\frac{\sqrt{1 - e^2} \sin v}{e + \cos v}\right) - e \frac{\sqrt{1 - e^2} \sin v}{1 + e \cos v} \qquad \text{(C4)}$$

Because of the transcendental form of equation (C4), explicit expressions for $\sin v$ and $\cos v$ in terms of a, e, and t_p cannot be obtained. However, the required partial derivatives may be obtained according to the following relations if v is regarded as an auxiliary variable:

$$\frac{\partial x}{\partial s} = \left[\frac{\partial x}{\partial s}\right] + \frac{\partial x}{\partial v}\frac{\partial v}{\partial s}, \qquad x \rightarrow y,z \qquad \text{(C5)}$$

for s equal to a, e, t_p where $[\partial x/\partial s]$ in brackets indicates that the derivative is now taken only so far as s appears explicitly in equations (C3). The derivatives from equations (C3) are

$$\frac{\partial x}{\partial v} = - \frac{a(1 - e^2)}{(1 + e \cos v)^2}\left[\alpha_1 \sin v - \alpha_2(e + \cos v)\right], \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \qquad \text{(C6)}$$

$$\left[\frac{\partial x}{\partial a}\right] = \frac{1 - e^2}{1 + e \cos v} (\alpha_1 \cos v + \alpha_2 \sin v), \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \qquad \text{(C7)}$$

$$\left[\frac{\partial x}{\partial e}\right] = -\frac{a[2e + (\cos v)(1 + e^2)]}{(1 + e \cos v)^2} (\alpha_1 \cos v + \alpha_2 \sin v), \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \tag{C8}$$

$$\left[\frac{\partial x}{\partial t_p}\right] = 0, \qquad x \rightarrow y,z \tag{C9}$$

and from equation (C4) the derivatives are

$$\frac{\partial v}{\partial a} = -\frac{3}{2}\sqrt{\frac{\mu}{a^5(1 - e^2)^3}} (1 + e \cos v)^2(t - t_p) \tag{C10}$$

$$\frac{\partial v}{\partial e} = \frac{\sin v}{1 - e^2} (2 + e \cos v) \tag{C11}$$

$$\frac{\partial v}{\partial t_p} = -\sqrt{\frac{\mu}{a^3(1 - e^2)^3}} (1 + e \cos v)^2 \tag{C12}$$

Combining equations (C6) to (C12) according to equation (C5) yields

$$\frac{\partial x}{\partial a} = \frac{1 - e^2}{1 + e \cos v} (\alpha_1 \cos v + \alpha_2 \sin v) + \frac{3}{2}\sqrt{\frac{\mu}{a^3(1 - e^2)}} (t - t_p)\Big[\alpha_1 \sin v - \alpha_2(e + \cos v)\Big], \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \tag{C13a}$$

$$\frac{\partial x}{\partial e} = -\frac{a\big[2e + (\cos v)(1 + e^2)\big]}{(1 + e \cos v)^2} (\alpha_1 \cos v + \alpha_2 \sin v) - a \sin v \frac{2 + e \cos v}{(1 + e \cos v)^2}\Big[\alpha_1 \sin v - \alpha_2(e + \cos v)\Big], \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \tag{C13b}$$

$$\frac{\partial x}{\partial t_p} = \sqrt{\frac{\mu}{a(1 - e^2)}}\Big[\alpha_1 \sin v - \alpha_2(e + \cos v)\Big], \qquad x,\alpha \rightarrow y,\beta;\ z,\gamma \tag{C13c}$$

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The desired results for D_a, D_e, and D_{t_p} are obtained by combining equations (C1) with equations (C13a), (C13b), and (C13c), respectively, according to equation (13a). Note that the following common factors in the process have the values shown:

Coefficients of terms containing R:

$$(\alpha_1 \cos v + \alpha_2 \sin v)^2 + (\beta_1 \cos v + \beta_2 \sin v)^2 + (\gamma_1 \cos v + \gamma_2 \sin v)^2 = 1$$

$$\begin{aligned}(\alpha_1 \cos v + \alpha_2 \sin v)&\left[\alpha_1 \sin v - \alpha_2(e + \cos v)\right] \\ &+ (\beta_1 \cos v + \beta_2 \sin v)\left[\beta_1 \sin v - \beta_2(e + \cos v)\right] \\ &+ (\gamma_1 \cos v + \gamma_2 \sin v)\left[\gamma_1 \sin v - \gamma_2(e + \cos v)\right] = -e \sin v\end{aligned}$$

Coefficients of terms containing C:

$$\begin{aligned}(\alpha_2 \cos v - \alpha_1 \sin v)&(\alpha_1 \cos v + \alpha_2 \sin v) \\ &+ (\beta_2 \cos v - \beta_1 \sin v)(\beta_1 \cos v + \beta_2 \sin v) \\ &+ (\gamma_2 \cos v - \gamma_1 \sin v)(\gamma_1 \cos v + \gamma_2 \sin v) = 0\end{aligned}$$

$$\begin{aligned}(\alpha_2 \cos v - \alpha_1 \sin v)&\left[\alpha_1 \sin v - \alpha_2(e + \cos v)\right] \\ &+ (\beta_2 \cos v - \beta_1 \sin v)\left[\beta_1 \sin v - \beta_2(e + \cos v)\right] \\ &+ (\gamma_2 \cos v - \gamma_1 \sin v)\left[\gamma_1 \sin v - \gamma_2(e + \cos v)\right] = -(1 + e \cos v)\end{aligned}$$

Coefficients of terms containing W:

$$\begin{aligned}(\alpha_1 \cos v + \alpha_2 \sin v)&\sin \Omega \sin I - \cos \Omega \sin I \\ &\cdot (\beta_1 \cos v + \beta_2 \sin v) + (\gamma_1 \cos v + \gamma_2 \sin v)\cos I = 0\end{aligned}$$

$$\begin{aligned}\left[\alpha_1 \sin v - \alpha_2(e + \cos v)\right]&\sin \Omega \sin I - \cos \Omega \sin I \\ &\cdot \left[\beta_1 \sin v - \beta_2(e + \cos v)\right] + \left[\gamma_1 \sin v - \gamma_2(e + \cos v)\right] \cos I = 0\end{aligned}$$

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The simplified results then are obtained as

$$D_a = \left[\frac{1 - e^2}{1 + e \cos v} - \frac{3}{2}\sqrt{\frac{\mu}{a^3(1 - e^2)}}\,(t - t_p)e \sin v\right]R$$

$$- \left[\frac{3}{2}\sqrt{\frac{\mu}{a^3(1 - e^2)}}\,(t - t_p)(1 + e \cos v)\right]C \qquad \text{(C14)}$$

$$D_e = (-a \cos v)R + a(\sin v)\left(\frac{2 + e \cos v}{1 + e \cos v}\right)C \qquad \text{(C15)}$$

$$D_{t_p} = -\sqrt{\frac{\mu}{a(1 - e^2)}}\,[(e \sin v)R + (1 + e \cos v)C] \qquad \text{(C16)}$$

Determination of D_ω, D_Ω, D_I

The disturbing functions of ω, Ω, and I are obtained by combining the derivatives of equations (C3) with equations (C1) according to equation (13a) after equations (C2) are used to eliminate α, β, and γ in equations (C1) and (C3). As this work is rather long but very direct, it is omitted. Results are:

$$D_\omega = Cr \qquad \text{(C17)}$$

$$D_I = Wr \sin u \qquad \text{(C18)}$$

$$D_\Omega = Cr \cos I - Wr \cos u \sin I \qquad \text{(C19)}$$

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TABLE I. - EQUATIONS FOR THE TWO-BODY ORBIT

(a) Classification of orbits[a]

Type of orbit	Energy, E_g	Angular momentum, h	Eccentricity, e	Semimajor axis, a	Semilatus rectum, p
Circle	$E_g = -\mu/2r$	$h = rV$	$e = 0$	$a = r$	$p = r$
Ellipse	< 0	$\neq 0$	< 1	> 0	> 0
Degenerate ellipse	< 0	$= 0$	$= 1$	> 0	$= 0$
Parabola	$= 0$	$\neq 0$	$= 1$	$= \infty$	> 0
Degenerate parabola	$= 0$	$= 0$	$= 1$	$= \infty$	$= 0$
Hyperbola	> 0	$\neq 0$	> 1	< 0	> 0
Degenerate hyperbola	> 0	$= 0$	$= 1$	< 0	$= 0$

[a]The degenerate conics included for completeness are straight lines.

The two-body equations describing motion on a conic section may be written in myriad forms. The following particularly useful equations have been selected for tabulation without derivation.

It is assumed that e and r are never negative. Special cases and restrictions necessary for real equations are indicated. It is also obvious that many of the equations become indeterminate or unbounded for certain conditions. (Symbols are defined in appendix A.)

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TABLE I. - Continued. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Two-body equations

Semimajor axis	
I-1:	$a = \frac{p}{1 - e^2}$
I-2:	$= \frac{r(1 + e \cos v)}{1 - e^2}$
I-3:	$= \frac{r}{1 - e \cos E}, \quad e < 1$
I-4:	$= \frac{r}{1 - e \cosh F}, \quad e > 1$
I-5:	$= \frac{h^2}{\mu(1 - e^2)}$
I-6:	$= \frac{1}{(2/r) - (V^2/\mu)}$
I-7:	$= \sqrt[3]{\frac{\mu}{n^2}}, \quad e < 1$
I-8:	$= \frac{r_p + r_a}{2}$
Semiminor axis	
I-9:	$b = \sqrt{ap}$
Eccentricity	
I-10:	$e = \sqrt{1 - \frac{p}{a}}$
I-11:	$= \sqrt{1 - \frac{h^2}{\mu a}}$
I-12:	$= \sqrt{1 + \frac{h^2}{\mu}\left(\frac{V^2}{\mu} - \frac{2}{r}\right)}$
I-13:	$= \sqrt{1 + p\left(\frac{V^2}{\mu} - \frac{2}{r}\right)}$
I-14:	$= \frac{1}{\cos v}\left(\frac{p}{r} - 1\right)$
I-15:	$= \frac{r_a - r_p}{r_a + r_p}$
Eccentric anomaly	
I-16:	$E = iF$
I-17:	$F = -iE$
I-18:	$\sin E = \frac{\sqrt{1 - e^2} \sin v}{1 + e \cos v}, \quad e < 1$
Eccentric anomaly (concluded)	
I-19:	$= \frac{r}{a} \frac{\sin v}{\sqrt{1 - e^2}}, \quad e < 1$
I-20:	$\sinh F = -\frac{r}{a} \frac{\sin v}{\sqrt{e^2 - 1}}, \quad e > 1$
I-21:	$\cos E = \frac{e + \cos v}{1 + e \cos v}, \quad e < 1$
I-22:	$= \frac{r}{a} \cos v + e, \quad e < 1$
I-23:	$\cosh F = \frac{r}{a} \cos v + e, \quad e > 1$
I-24:	$\tan E = \frac{\sqrt{1 - e^2} \sin v}{e + \cos v}, \quad e < 1$
I-25:	$\tan \frac{E}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{v}{2}, \quad e < 1$
I-26:	$\tanh \frac{F}{2} = \sqrt{\frac{e - 1}{e + 1}} \tan \frac{v}{2}, \quad e > 1$
Energy per unit mass	
I-27:	$Eg = \frac{V^2}{2} - \frac{\mu}{r}$
I-28:	$= -\frac{\mu}{2p}(1 - e^2)$
I-29:	$= -\frac{\mu}{2a}$
I-30:	$= -\frac{\mu}{2r}, \quad e = 0$
Angular momentum per unit mass, or twice rate of description of area	
I-31:	$h_x = y\dot{z} - z\dot{y}$
I-32:	$h_y = z\dot{x} - x\dot{z}$
I-33:	$h_z = x\dot{y} - y\dot{x}$
I-34:	$h^2 = h_x^2 + h_y^2 + h_z^2$
I-35:	$= r^4\dot{u}^2$
I-36:	$= \mu p$
I-37:	$= \mu a(1 - e^2)$
I-38:	$= \mu r, \quad e = 0$
I-39:	$h = h_z, \quad I = 0 = \Omega$
I-40:	$= r_p V_p$
I-41:	$= rV \cos \psi$

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TABLE I. - Continued. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Continued. Two-body equations

Orbital inclination

I-42:	$\tan I = \frac{\sqrt{h_x^2 + h_y^2}}{h_z}, \quad 0 \leq I \leq \pi$
I-43:	$\sin I = \frac{\sqrt{h_x^2 - h_y^2}}{h}$
I-44:	$\cos I = \frac{h_z}{h}$

Mean anomaly

I-45:	$M = n(t - t_p), \quad e < 1$
I-46:	$= \nu(t - t_p), \quad e > 1$
I-47:	$= E - e \sin E, \quad e < 1$
I-48:	$= -F + e \sinh F, \quad e > 1$
I-49:	$\dot{M} = n$

Mean angular motion

I-50:	$n = \frac{M}{t - t_p}, \quad e < 1$
I-51:	$= \frac{2\pi}{P}, \quad e < 1$
I-52:	$= \sqrt{\frac{\mu}{a^3}} \quad e < 1$
I-53:	$= \sqrt{\frac{\mu}{p^3}(1 - e^2)^3}, \quad e < 1$
I-54:	$\nu = \sqrt{\frac{\mu}{p^3}(e^2 - 1)^3}, \quad e > 1$
I-55:	$= -in$

Semilatus rectum

I-56:	$p = a(1 - e^2)$
I-57:	$= r(1 + e \cos v)$
I-58:	$= h^2/\mu$
I-59:	$= r^4\dot{v}^2/\mu$
I-60:	$= \frac{\mu}{V^2}(1 + e^2 + 2e \cos v)$
I-61:	$= \frac{r(1 - e^2)}{1 - e \cos E}, \quad e < 1$
I-62:	$= \frac{\mu(1 - e^2)}{2(\mu/r) - V^2}$
I-63:	$= \sqrt[3]{\frac{\mu}{n^2}}(1 - e^2), \quad e < 1$

Semilatus rectum (concluded)

I-64:	$p = \frac{2r_p r_a}{r_p + r_a}$
I-65:	$= r(1 + \cos v), \quad e = 1$
I-66:	$= 2r_p, \quad e = 1$
I-67:	$= r, \quad e = 0$

Period

I-68:	$P = \frac{2\pi}{n}, \quad e < 1$
I-69:	$= 2\pi\sqrt{\frac{a^3}{\mu}}, \quad e < 1$

Radius

I-70:	$r = \frac{p}{1 + e \cos v}$
I-71:	$= \sqrt{x^2 + y^2 + z^2}$
I-72:	$= \frac{a(1 - e^2)}{1 + e \cos v}$
I-73:	$= a(1 - e \cos E), \quad e < 1$
I-74:	$= a(1 - e \cosh F), \quad e > 1$
I-75:	$= r_p\left(1 + \tan^2 \frac{v}{2}\right), \quad e = 1$
I-76:	$= p, \quad e = 0$
I-77:	$\dot{r} = V \sin \psi$
I-78:	$= \frac{x\dot{x} + y\dot{y} + z\dot{z}}{r}$
I-79:	$= \frac{x\dot{x} + y\dot{y}}{r}, \quad I = 0 = \Omega$
I-80:	$= \sqrt{\frac{\mu}{p}}\, e \sin v$
I-81:	$\ddot{r} = \frac{\mu e \cos v}{r^2}$

Radius at pericenter

I-82:	$r_p = \frac{p}{1 + e}$
I-83:	$= 2a - r_a$
I-84:	$= a(1 - e)$
I-85:	$= \frac{p}{2}, \quad e = 1$

Radius at apocenter

I-86:	$r_a = \frac{p}{1 - e}$
I-87:	$= 2a - r_p$
I-88:	$= a(1 + e)$

TABLE I. - Continued. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Continued. Two-body equations

Range (of ballistic missile) on sphere of radius r_s intersected by an elliptical orbit

I-89: $$Rg = 2r_s(\pi - v_s), \qquad 0 \leq v_s \leq \pi$$

I-90: $$= 2r_s\left\{\pi - \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_s} - 1\right)\right]\right\}$$

Time of pericenter passage

I-91: $$t_p = t - \frac{M}{n}, \qquad e < 1$$

I-92: $$= t - \frac{E - e \sin E}{n}, \qquad e < 1$$

I-93: $$= t + \frac{F - e \sinh F}{\nu}, \qquad e > 1$$

I-94: $$= t - \sqrt{\frac{p^3}{\mu}} \int_0^v \frac{dv}{(1 + e \cos v)^2}$$

I-95: $$= t - \sqrt{\frac{p^3}{\mu}} \left\{\frac{1}{1 - e^2}\left[\int_0^v \frac{dv}{1 + e \cos v} - \frac{e \sin v}{1 + e \cos v}\right]\right\}$$

I-96: $$= t - \sqrt{\frac{p^3}{\mu}} \frac{1}{1 - e^2}\left(\frac{E}{\sqrt{1 - e^2}} - \frac{e \sin v}{1 + e \cos v}\right), \qquad e < 1$$

I-97: $$= t - \sqrt{\frac{p^3}{\mu}} \frac{1}{1 - e^2}\left(\frac{F}{\sqrt{e^2 - 1}} - \frac{e \sin v}{1 + e \cos v}\right), \qquad e > 1$$

I-98: $$= t - \sqrt{\frac{p^3}{\mu}} \frac{1}{1 - e^2}\left[\frac{2}{\sqrt{1 - e^2}} \tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}} \tan \frac{v}{2}\right) - \frac{e \sin v}{1 + e \cos v}\right], \qquad e < 1$$

I-99: $$= t - \sqrt{\frac{p^3}{\mu}} \frac{1}{1 - e^2}\left[\frac{1}{\sqrt{e^2 - 1}} \log\left(\frac{\sqrt{e + 1} + \sqrt{e - 1} \tan v/2}{\sqrt{e + 1} - \sqrt{e - 1} \tan v/2}\right) - \frac{e \sin v}{1 + e \cos v}\right], \qquad e > 1$$

I-100: $$= t - \sqrt{\frac{p^3}{\mu}} \frac{1}{3}\left[\frac{(2 + \cos v)}{(1 + \cos v)^2} \sin v\right], \qquad e = 1$$

I-101: $$= t - \frac{r^2}{2\mu - rV^2}\left\{\frac{2\mu}{\sqrt{r(2\mu - rV^2)}} \tan^{-1}\left[\frac{\sqrt{\mu^2 r - h^2(2\mu - rV^2)} + \sqrt{r}(\mu - rV^2)}{r\dot{r}\sqrt{2\mu - rV^2}}\right] - \dot{r}\right\}, \qquad e < 1$$

I-102: $$= t - \frac{r^2}{2\mu - rV^2}\left\{\frac{\mu}{\sqrt{r(rV^2 - 2\mu)}} \log\left[\frac{r\dot{r}\sqrt{rV^2 - 2\mu} + \sqrt{r}(rV^2 - \mu) - \sqrt{\mu^2 r - h^2(2\mu - rV^2)}}{r\dot{r}\sqrt{rV^2 - 2\mu} - \sqrt{r}(rV^2 - \mu) + \sqrt{\mu^2 r - h^2(2\mu - rV^2)}}\right] - \dot{r}\right\}, \qquad e > 1$$

I-103: $$= t - \frac{r}{3\dot{r}}\left(2 - \frac{h^2}{\mu r}\right)\left(1 + \frac{h^2}{\mu r}\right), \qquad e = 1$$

TABLE I. - Continued. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Continued. Two-body equations

Argument of latitude (same as polar angle)

I-104: $u = v + \omega$

I-105: $\tan u = \frac{y}{x}, \qquad I = 0 = \Omega$

I-106: $= \frac{z \sin I + (y \cos \Omega - x \sin \Omega)\cos I}{x \cos \Omega + y \sin \Omega}$

I-107: $\dot{u} = \frac{\sqrt{\mu p}}{r^2}$

I-108: $= \frac{V \cos \psi}{r}$

True anomaly

I-109: $v = u - \omega$

I-110: $\tan v = \sqrt{\frac{p}{\mu}}\, \frac{r\dot{r}}{p - r}$

I-111: $= \frac{hr\dot{r}}{h^2 - \mu r}$

I-112: $= \frac{h(x\dot{x} + y\dot{y} + z\dot{z})}{h^2 - \mu r}$

I-113: $\sin v = \sqrt{\frac{p}{\mu}}\, \frac{\dot{r}}{e}$

I-114: $= \frac{\sqrt{1 - e^2} \sin E}{1 - e \cos E}, \qquad e < 1$

I-115: $= \frac{a}{r} \sqrt{1 - e^2} \sin E, \qquad e < 1$

I-116: $= \frac{-\sqrt{e^2 - 1} \sinh F}{1 - e \cosh F}, \qquad e > 1$

I-117: $\cos v = \frac{1}{e}\left(\frac{p}{r} - 1\right)$

I-118: $= \frac{\cos E - e}{1 - e \cos E}, \qquad e < 1$

I-119: $= \frac{a}{r} (\cos E - e), \qquad e < 1$

I-120: $= \frac{\cosh F - e}{1 - e \cosh F}, \qquad e > 1$

True anomaly (concluded)

I-121: $\tan \frac{v}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}, \qquad e < 1$

I-122: $= \sqrt{\frac{e + 1}{e - 1}} \tanh \frac{F}{2}, \qquad e > 1$

I-123: $\tan v = Y/X$

I-124: $\dot{v} = \frac{\sqrt{\mu p}}{r^2}$

I-125: $= h/r^2$

I-126: $= \dot{u}$

I-127: $= \frac{V \cos \psi}{r}$

Velocity

I-128: $V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

I-129: $= \dot{r}^2 + r^2\dot{u}^2$

I-130: $= \dot{r}^2 + r^2\dot{v}^2$

I-131: $= \dot{r}^2 + \frac{\mu p}{r^2}$

I-132: $= \frac{\mu}{p} (1 + e^2 + 2e \cos v)$

I-133: $= \mu\left(\frac{2}{r} - \frac{1 - e^2}{p}\right)$

I-134: $= \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

I-135: $= \frac{\mu}{r}, \qquad e = 0$

I-136: $V_p^2 = \frac{\mu}{p} (1 + e)^2$

I-137: $= \frac{r_a}{r_p} \frac{2\mu}{r_a + r_p}$

I-138: $V_a^2 = \frac{\mu}{p} (1 - e)^2$

I-139: $= \frac{r_p}{r_a} \frac{2\mu}{r_a + r_p}$

TABLE I. - Continued. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Continued. Two-body equations

Rectangular coordinates

I-140:	$x = r(\cos u \cos \Omega - \sin u \sin \Omega \cos I)$
I-141:	$= r \cos u, \quad I = 0 = \Omega$
I-142:	$X = r \cos v$
I-143:	$= x(\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos I) + y(\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos I) + z \sin \omega \sin I$
I-144:	$= a(\cos E - e), \quad e < 1$
I-145:	$= a(\cosh F - e), \quad e > 1$
I-146:	$= r_p\left(1 - \tan^2 \frac{v}{2}\right), \quad e = 1$
I-147:	$y = r(\cos u \sin \Omega + \sin u \cos \Omega \cos I)$
I-148:	$= r \sin u, \quad I = 0 = \Omega$
I-149:	$Y = r \sin v$
I-150:	$= -x(\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos I) - y(\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos I) + y \cos \omega \sin I$
I-151:	$= a\sqrt{1 - e^2} \sin E, \quad e < 1$
I-152:	$= -a\sqrt{e^2 - 1} \sinh F, \quad e > 1$
I-153:	$= 2r_p \tan \frac{v}{2}, \quad e = 1$
I-154:	$z = r \sin u \sin I$

Velocity components

I-155:	$\dot{x} = -\sqrt{\frac{\mu}{p}}\left[\sin \Omega \cos I(e \cos \omega + \cos u) + \cos \Omega(e \sin \omega + \sin u)\right]$
I-156:	$= -\sqrt{\frac{\mu}{p}}\,(e \sin \omega + \sin u), \quad I = 0 = \Omega$
I-157:	$\dot{X} = -\sqrt{\frac{\mu}{p}} \sin v$
I-158:	$= -a\dot{E} \sin E, \quad e < 1$
I-159:	$= a\dot{F} \sinh F, \quad e > 1$
I-160:	$\dot{y} = \sqrt{\frac{\mu}{p}}\left[\cos \Omega \cos I(e \cos \omega + \cos u) - \sin \Omega(e \sin \omega + \sin u)\right]$
I-161:	$= \sqrt{\frac{\mu}{p}}\,(e \cos \omega + \cos u), \quad I = 0 = \Omega$
I-162:	$\dot{Y} = \sqrt{\frac{\mu}{p}}\,(e + \cos v)$
I-163:	$= a\sqrt{1 - e^2}\, \dot{E} \cos E, \quad e < 1$
I-164:	$= -a\sqrt{e^2 - 1}\, \dot{F} \cosh F, \quad e > 1$
I-165:	$\dot{z} = \sqrt{\frac{\mu}{p}} \sin I(e \cos \omega + \cos u)$

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TABLE I. - Concluded. EQUATIONS FOR THE TWO-BODY ORBIT

(b) Concluded. Two-body equations

	Path angle
I-166:	$\tan\psi = \dfrac{\dot{r}}{r\dot{v}}$
I-167:	$= \dfrac{x\dot{x} + y\dot{y} + z\dot{z}}{h}$
I-168:	$= \dfrac{e \sin v}{1 + e \cos v}$
I-169:	$= \dfrac{er}{p} \sin v$
I-170:	$\sin\psi = \dfrac{\dot{r}}{V}$
I-171:	$= \dfrac{x\dot{x} + y\dot{y} + z\dot{z}}{rV}$
I-172:	$= \dfrac{e \sin v}{\sqrt{1 + e^2 + 2e \cos v}}$
I-173:	$\cos\psi = \dfrac{r\dot{v}}{V}$
I-174:	$= \dfrac{h}{rV}$
I-175:	$= \dfrac{1 + e \cos v}{\sqrt{1 + e^2 + 2e \cos v}}$
	Argument of pericenter
I-176:	$\omega = u - v$
	Ascending node
I-177:	$\tan\Omega = \dfrac{h_x}{-h_y}$
I-178:	$\sin\Omega = \dfrac{h_x}{\sqrt{h_x^2 + h_y^2}}$
I-179:	$\cos\Omega = \dfrac{-h_y}{\sqrt{h_x^2 + h_y^2}}$

TABLE II. - DERIVATIVES OF ORBITAL ELEMENTS AND PARAMETERS[a]

(a) Time derivatives of orbital elements due to perturbations

Semilatus rectum

$$\dot{p} = \left(2r\sqrt{\frac{p}{\mu}}\right)C$$

$$= 2\,\frac{p}{V}\left[T + \left(\frac{r}{p}\,e\sin v\right)N\right]$$

$$= 2\,\frac{r}{V}\,T, \qquad e = 0$$

$$= 2\,\frac{r}{V}\,C, \qquad e = 0$$

Eccentricity

$$\dot{e} = \sqrt{\frac{p}{\mu}}\left\{(\sin v)R + \frac{r}{p}\left[2\cos v + e(1 + \cos^2 v)\right]C\right\}$$

$$= \frac{1}{V}\left[2(e + \cos v)T - \left(\frac{r}{a}\sin v\right)N\right]$$

$$= \pm\frac{1}{V}\sqrt{4T^2 + N^2}, \qquad e = 0$$

$$= \pm\frac{1}{V}\sqrt{4C^2 + R^2}, \qquad e = 0$$

Argument of pericenter

$$\dot{\omega} = \sqrt{\frac{p}{\mu}}\left[\frac{\sin v}{e}\left(1 + \frac{r}{p}\right)C - \left(\frac{\cos v}{e}\right)R - \left(\frac{r}{p}\sin u\cot I\right)W\right], \qquad e \neq 0$$

$$= \frac{1}{V}\left[\frac{2\sin v}{e}\,T + \frac{r}{pe}\,(2e + \cos v + e^2\cos v)N - \left(\frac{rV}{\sqrt{p\mu}}\sin u\cot I\right)W\right], \qquad e \neq 0$$

$$= \frac{V}{2r} + \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2} + \frac{1}{V}\left[N\,\frac{2T^2 + N^2}{4T^2 + N^2} - (\sin u\cot I)W\right], \qquad e = 0$$

$$= \frac{V}{2r} - \frac{C\dot{R} - R\dot{C}}{4C^2 + R^2} - \frac{1}{V}\left[\frac{RC^2}{4C^2 + R^2} + (\sin u\cot I)W\right], \qquad e = 0$$

[a] $T\dot{N} - N\dot{T} = 0$ if T/N is constant, and $C\dot{R} - R\dot{C} = 0$ if C/R is constant. However, T/N and C/R are never both constant at once.

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TABLE II. - Continued. DERIVATIVES OF ORBITAL ELEMENTS AND PARAMETERS

(a) Continued. Time derivatives of orbital elements due to perturbations

Time of pericenter

$$\dot{t}_p = \frac{1}{1 - e^2}\sqrt{\frac{p}{\mu}}\left\{\left[p\sqrt{\frac{p}{\mu}}\left(2\frac{r}{p} - \frac{\cos v}{e}\right) - 3(t - t_p)e \sin v\right]R + \left[p\sqrt{\frac{p}{\mu}}\frac{\sin v}{e}\left(1 + \frac{r}{p}\right) - 3(t - t_p)\frac{p}{r}\right]C\right\}, \qquad 0 \neq e \neq 1$$

$$= \frac{1}{V}\sqrt{\frac{p}{\mu}}\left\{\frac{1}{1 - e^2}\left[2r(\sin v)\left(e + \frac{p}{re}\right) - 3(t - t_p)V^2\sqrt{\frac{p}{\mu}}\right]T + \left(\frac{r \cos v}{e}\right)N\right\}, \qquad 0 \neq e \neq 1$$

$$= \frac{a^2}{\mu}\left\{\left[(\sin v)\left(\frac{2 + e^2}{e} - \frac{r}{a}\cos v\right) - 3\frac{p}{r}\frac{E}{\sqrt{1 - e^2}}\right]C - \left[2e \cos v + \frac{\cos v}{e} - 3 + \frac{r}{a} + 3e\frac{E \sin v}{\sqrt{1 - e^2}}\right]R\right\}, \qquad 0 < e < 1$$

$$= \frac{a}{V}\sqrt{\frac{p}{\mu}}\left\{\frac{1}{1 - e^2}\left[\left(\frac{2}{e} + 3e + \frac{e^2(e + \cos v)}{1 + e \cos v}\right)\sin v - 3\frac{p}{\mu}V^2\frac{E}{\sqrt{1 - e^2}}\right]T + \left(\frac{r}{ae}\cos v\right)N\right\}, \qquad 0 < e < 1$$

The results for $\dot{t}_p$ for $e > 1$ are identical to the preceding results for $0 < e < 1$, but with E replaced by iF and $\sqrt{1 - e^2}$ replaced by $i\sqrt{e^2 - 1}$.

$$\dot{t}_p = \frac{p^2}{5\mu(1 + \cos v)^2}\left\{\left[2(\sin v)(\cos^2 v + 3 \cos v + 1)\right]C - \left[2 \cos^3 v + 4 \cos^2 v + \cos v - 2\right]R\right\}, \qquad e = 1$$

$$= \frac{p}{V(1 + \cos v)}\sqrt{\frac{p}{\mu}}\left\{\left[\frac{(4 \cos^2 v + 7 \cos v + 4)\sin v}{5(1 + \cos v)}\right]T + (\cos v)N\right\}, \qquad e = 1$$

$$= \frac{1}{2} - \frac{r}{V^2}\left[\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right)N + 3vT\right] + \frac{r}{V}\left(\frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}\right), \qquad e = 0$$

$$= \frac{1}{2} + \frac{r}{V^2}\left[\left(\frac{7C^2 + 2R^2}{4C^2 + R^2}\right)R - 3vC\right] + \frac{r}{V}\left(\frac{R\dot{C} - C\dot{R}}{4C^2 + R^2}\right), \qquad e = 0$$

TABLE II. - Continued. DERIVATIVES OF ORBITAL ELEMENTS AND PARAMETERS

(a) Concluded. Time derivatives of orbital elements due to perturbations

Ascending node

$$\dot{\Omega} = \left(\frac{r}{\sqrt{p\mu}} \frac{\sin u}{\sin I}\right)W, \qquad I \neq 0$$

Inclination

$$\dot{I} = \left(\frac{r}{\sqrt{p\mu}} \cos u\right)W$$

Semimajor axis

$$\dot{a} = \frac{2a}{1 - e^2}\sqrt{\frac{p}{\mu}}\left[(e \sin v)R + \left(\frac{p}{r}\right)C\right]$$

$$= \left(\frac{2a^2V}{\mu}\right) T$$

$$= \frac{2r}{V} C, \qquad e = 0$$

$$= \left(\frac{2r}{V}\right) T, \qquad e = 0$$

Radius of pericenter

$$\dot{r}_p = \frac{r_p}{1 + e}\sqrt{\frac{p}{\mu}}\left\{\frac{r}{p}\left[e \sin^2 v + 2(1 - \cos v)\right]C - (\sin v)R\right\}$$

$$= \frac{1}{V}\left[2r_p \frac{1 - \cos v}{1 + e} T + (r \sin v)N\right]$$

$$= \frac{r}{V}\left(2C \mp \sqrt{4C^2 + R^2}\right), \qquad e = 0$$

$$= \frac{r}{V}\left(2T \mp \sqrt{4T^2 + N^2}\right), \qquad e = 0$$

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TABLE II. - Concluded. DERIVATIVES OF ORBITAL ELEMENTS AND PARAMETERS

(b) Time derivatives of orbital parameters due to both orbital motion and perturbations

Mean anomaly

$$\dot{M} = n + \sqrt{\frac{p(1 - e^2)}{\mu}}\left[\left(\frac{\cos v}{e} - 2\,\frac{r}{p}\right)R - \frac{\sin v}{e}\left(1 + \frac{r}{p}\right)C\right], \qquad 0 < e \lesssim 1$$

$$= n - \frac{\sqrt{1 - e^2}}{V}\left[2(\sin v)\left(\frac{re}{p} + \frac{1}{e}\right)T + \left(\frac{r \cos v}{ae}\right)N\right], \qquad 0 < e \lesssim 1$$

The results for $\dot{M}$ for $e > 1$ are identical to the preceding results for $0 < e \lesssim 1$, but with n replaced by $i\nu$ and $\sqrt{1 - e^2}$ replaced by $i\sqrt{e^2 - 1}$.

$$\dot{M} = n - \frac{V}{2r} + \frac{N}{V}\left(\frac{6T^2 + N^2}{4T^2 + N^2}\right) \pm \frac{2TN}{V\sqrt{4T^2 + N^2}} - \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}, \qquad e = 0$$

$$= n - \frac{V}{2r} - \frac{R}{V}\left(\frac{7C^2 + 2R^2}{4C^2 + R^2}\right) \mp \frac{2CR}{V\sqrt{4C^2 + R^2}} + \left(\frac{C\dot{R} - R\dot{C}}{4C^2 + R^2}\right), \qquad e = 0$$

True anomaly

$$\dot{v} = \frac{\sqrt{\mu p}}{r^2} + \frac{1}{e}\sqrt{\frac{p}{\mu}}\left[(\cos v)R - (\sin v)\left(1 + \frac{r}{p}\right)C\right], \qquad e \neq 0$$

$$= \frac{\sqrt{\mu p}}{r^2} - \frac{1}{Ve}\left[(2 \sin v)T + \frac{r}{p}\,(2e + e^2 \cos v + \cos v)N\right], \qquad e \neq 0$$

$$= \frac{V}{2r} - \frac{N}{V}\left(\frac{2T^2 + N^2}{4T^2 + N^2}\right) - \frac{T\dot{N} - N\dot{T}}{4T^2 + N^2}, \qquad e = 0$$

$$= \frac{V}{2r} + \frac{R}{V}\left(\frac{C^2}{4C^2 + R^2}\right) + \frac{C\dot{R} - R\dot{C}}{4C^2 + R^2}, \qquad e = 0$$

TABLE III. - COMPONENTS OF THE DISTURBING ACCELERATION

$$R = A_x(\cos u \cos \Omega - \sin u \sin \Omega \cos I) + A_y(\cos u \sin \Omega + \sin u \cos \Omega \cos I) + A_z \sin u \sin I$$

$$C = A_x(-\sin u \cos \Omega - \cos u \sin \Omega \cos I) + A_y(-\sin u \sin \Omega + \cos u \cos \Omega \cos I) + A_z \cos u \sin I$$

$$W = A_x \sin \Omega \sin I - A_y \cos \Omega \sin I + A_z \cos I$$

$$T = (1 + e^2 + 2e \cos v)^{-(1/2)}[(1 + e \cos v)C + (e \sin v)R]$$

$$= \frac{1}{V}\left(\frac{\sqrt{\mu p}}{r} C + \dot{r}R\right)$$

$$= C, \qquad e = 0$$

$$\dot{T} = \dot{C} + \frac{R^2}{V}, \qquad e = 0$$

$$N = (1 + e^2 + 2e \cos v)^{-(1/2)}[(e \sin v)C - (1 + e \cos v)R]$$

$$= \frac{1}{V}\left(\dot{r}C - \frac{\sqrt{\mu p}}{r} R\right)$$

$$= -R, \qquad e = 0$$

$$\dot{N} = \frac{CR}{V} - \dot{R}, \qquad e = 0$$

$$C = (1 + e^2 + 2e \cos v)^{-(1/2)}[1 + e \cos v)T + (e \sin v)N]$$

$$= \frac{1}{V}\left(\frac{\sqrt{\mu p}}{r} T + \dot{r}N\right)$$

$$\dot{C} = \dot{T} - \frac{N^2}{V}, \qquad e = 0$$

$$R = (1 + e^2 + 2e \cos v)^{-(1/2)}[(e \sin v)T - (1 + e \cos v)N]$$

$$= \frac{1}{V}\left(\dot{r}T - \frac{\sqrt{\mu p}}{r} N\right)$$

$$\dot{R} = \frac{RT}{V} - \dot{N}, \qquad e = 0$$

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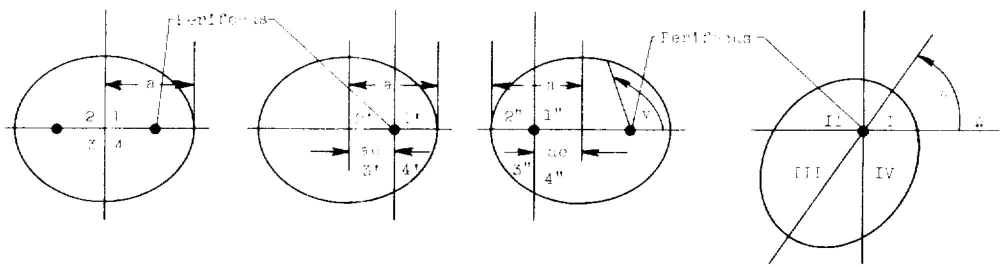
TABLE IV. - QUALITATIVE EFFECTS OF THE DISTURBING ACCELERATION COMPONENTS[a]



Derivative of element or parameter	Component				
	T	N	C	R	W
Semimajor axis, a	Always +	0	Always +	1' and 2', + 3' and 4', -	0
Semilatus rectum, p	Always +	1' and 2', + 3' and 4', -	Always +	0	0
Radius of pericenter, r_p	Always +	1' and 2', + 3' and 4', -	Always +	1' and 2', - 3' and 4', +	0
Eccentricity, e	1 and 4, +	1' and 2', -	$\cos v > \frac{\sqrt{1 - e^2} - 1}{e}$, +	1' and 2', +	
	2 and 3, -	3' and 4', +	$\cos v < \frac{\sqrt{1 - e^2} - 1}{e}$, -	3' and 4', -	0
Mean anomaly, M	1' and 2', -	1' and 4', -	1' and 2', -	$\cos v > \frac{\sqrt{1 + 8e^2} - 1}{2e}$, +	
	3' and 4', +	2' and 3', +	3' and 4', +	$\cos v < \frac{\sqrt{1 + 8e^2} - 1}{2e}$, -	0
True anomaly, v	1' and 2', - 3' and 4', +	1" and 4", - 2" and 3", +	1' and 2', - 3' and 4', +	1' and 4', + 2' and 3', -	0
Argument of pericenter, ω	1' and 2', + 3' and 4', +	1" and 4", + 2" and 3", +	1' and 2', + 3' and 4', +	1' and 4', - 2' and 3', -	I and II, - III and IV, +
Ascending node, Ω	0	0	0	0	I and II, + III and IV, -
Inclination, I	0	0	0	0	I and IV, + II and III, -

[a]The table shows intervals in the orbit when positive components of disturbing acceleration produce positive and negative derivatives of the orbital elements. Results given are valid for $e < 1$. Intervals are denoted by quadrant according to the sketches where convenient. In the case of the parameters M and v it must be remembered that the table applies only to perturbation terms of the derivative. Results are opposite in every case for negative disturbing components.

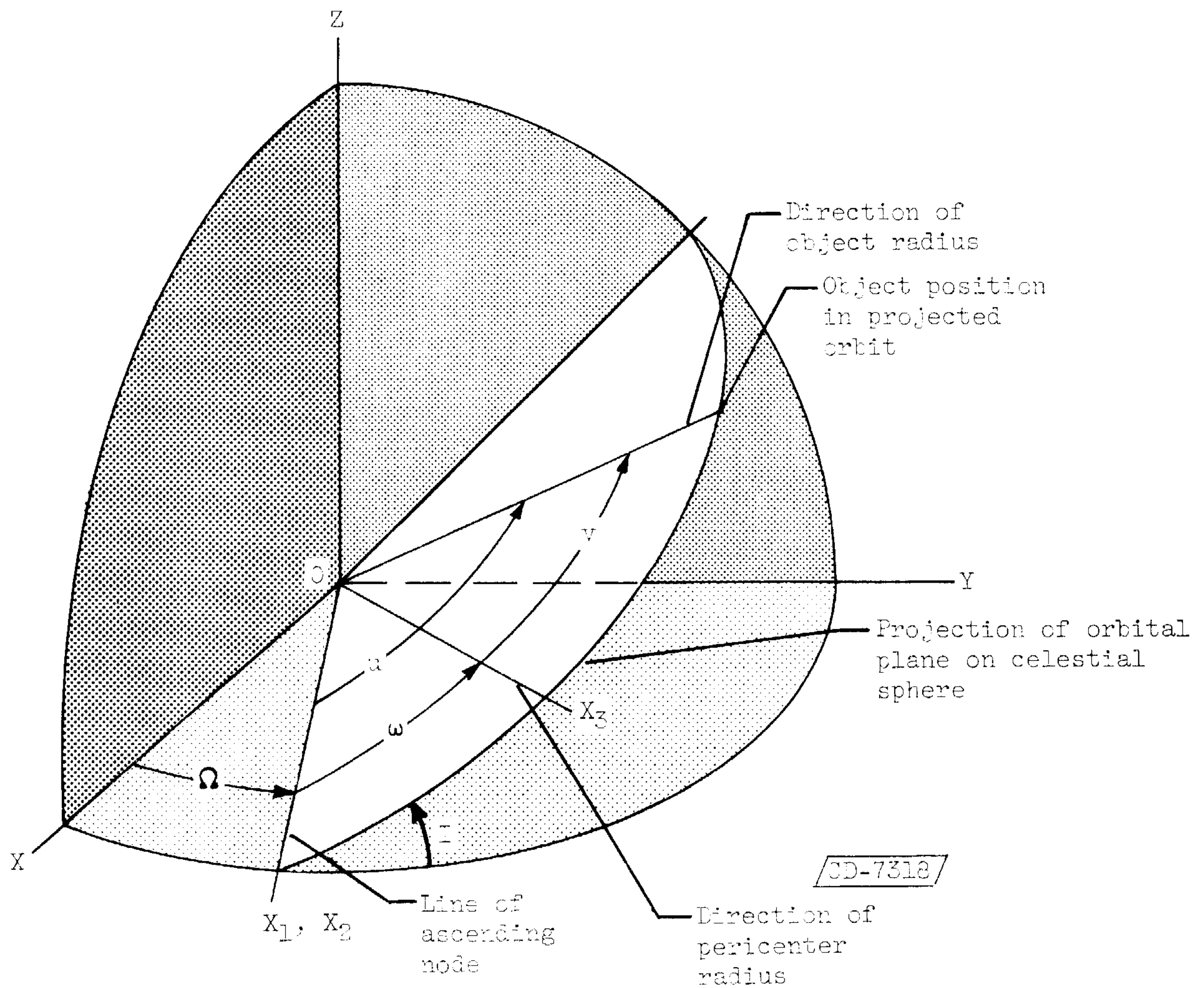


Figure 1. - Diagram showing various orbital parameters in general OX, OY, OZ system and location of OX_1, OX_2, and OX_3 coordinate axes in alternate systems.

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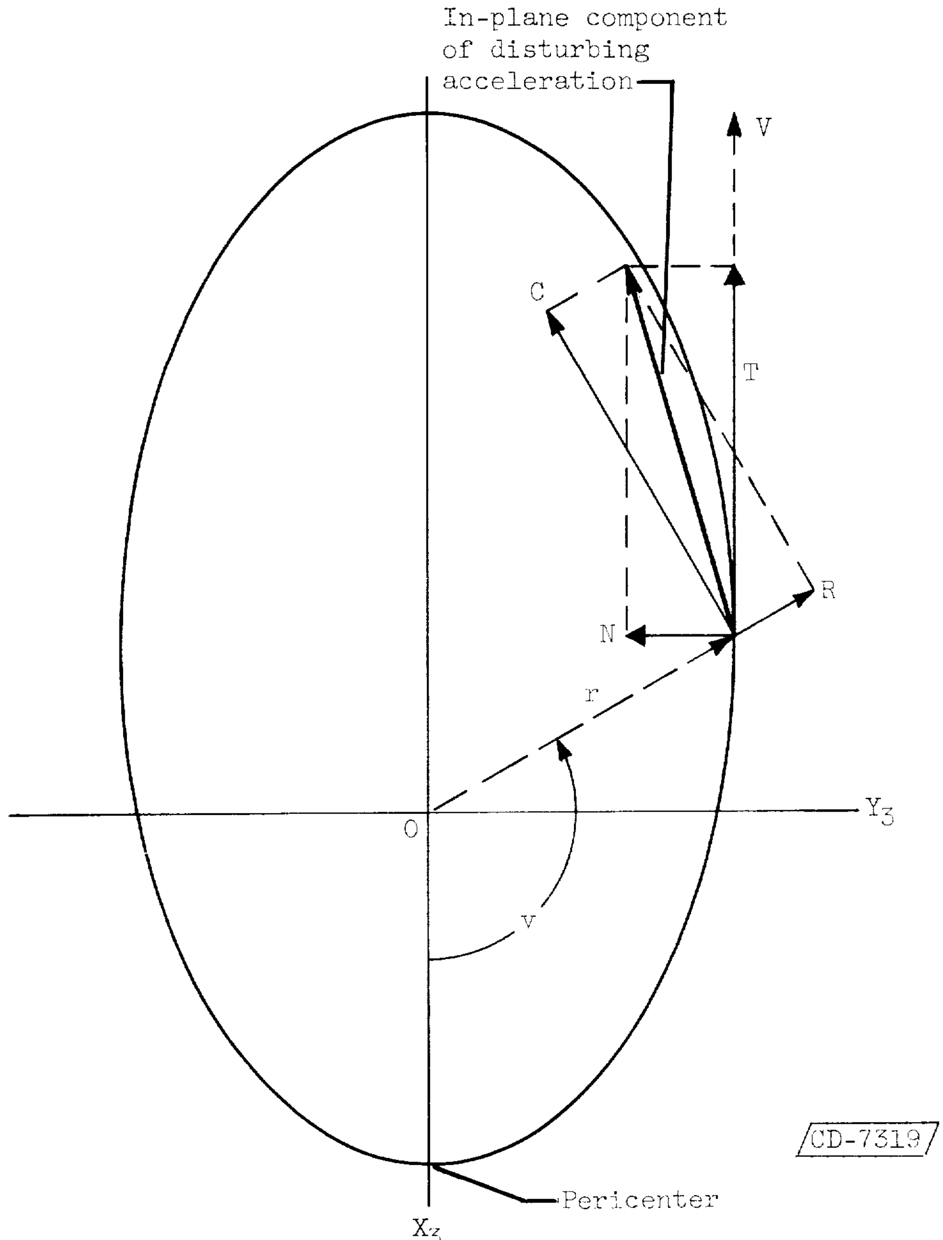

Figure 2. - Diagram in orbital plane showing resolution of in-plane disturbing acceleration into radial (R) and circumferential (C), or tangential (T) and normal (N) systems.